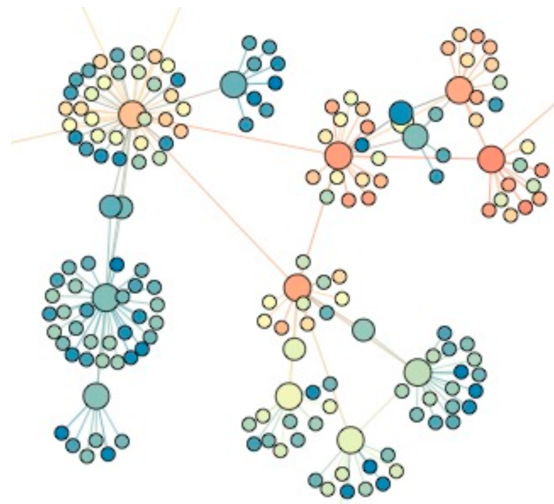

SNA 2C: Growth & Preferential Attachment Models

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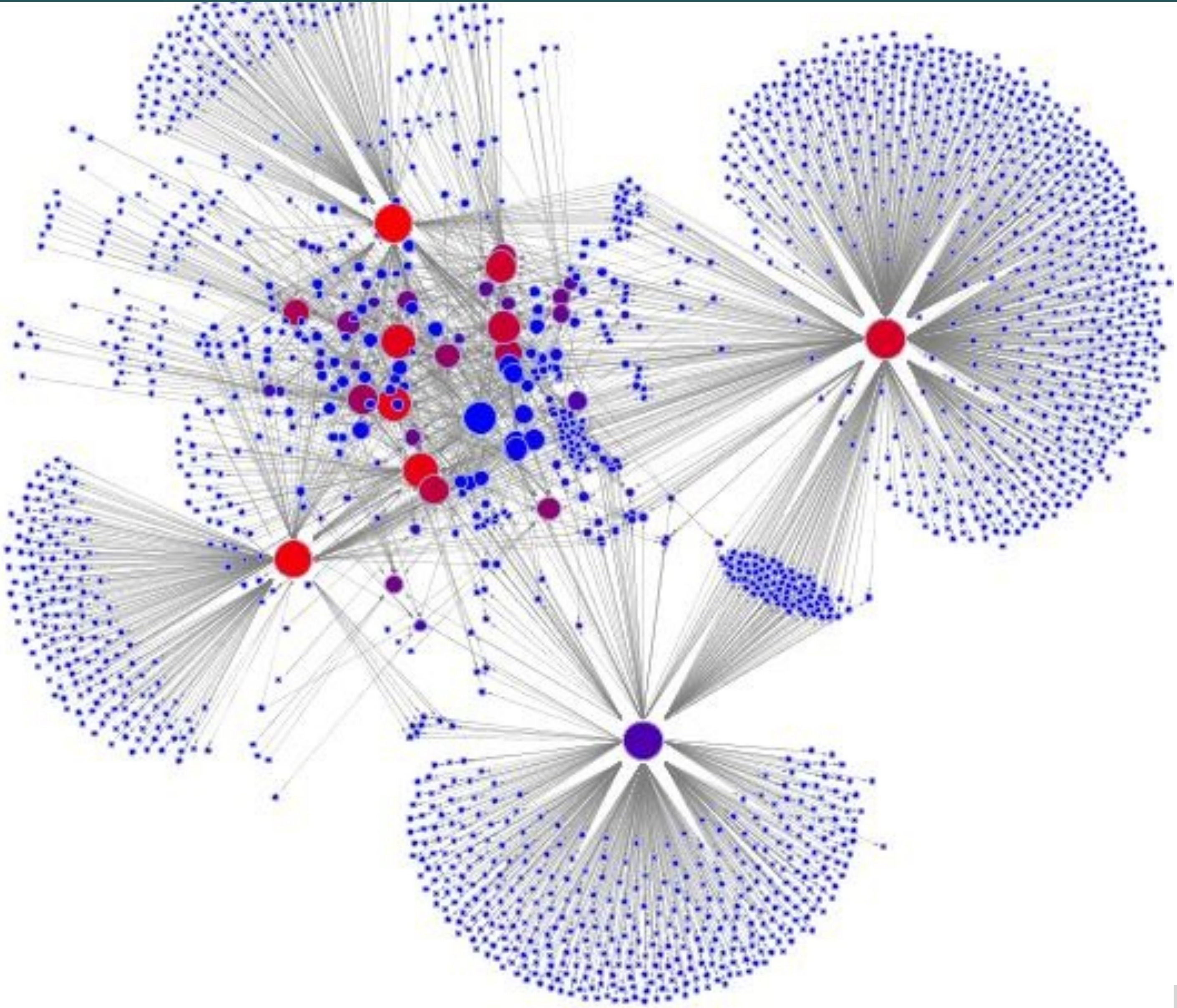
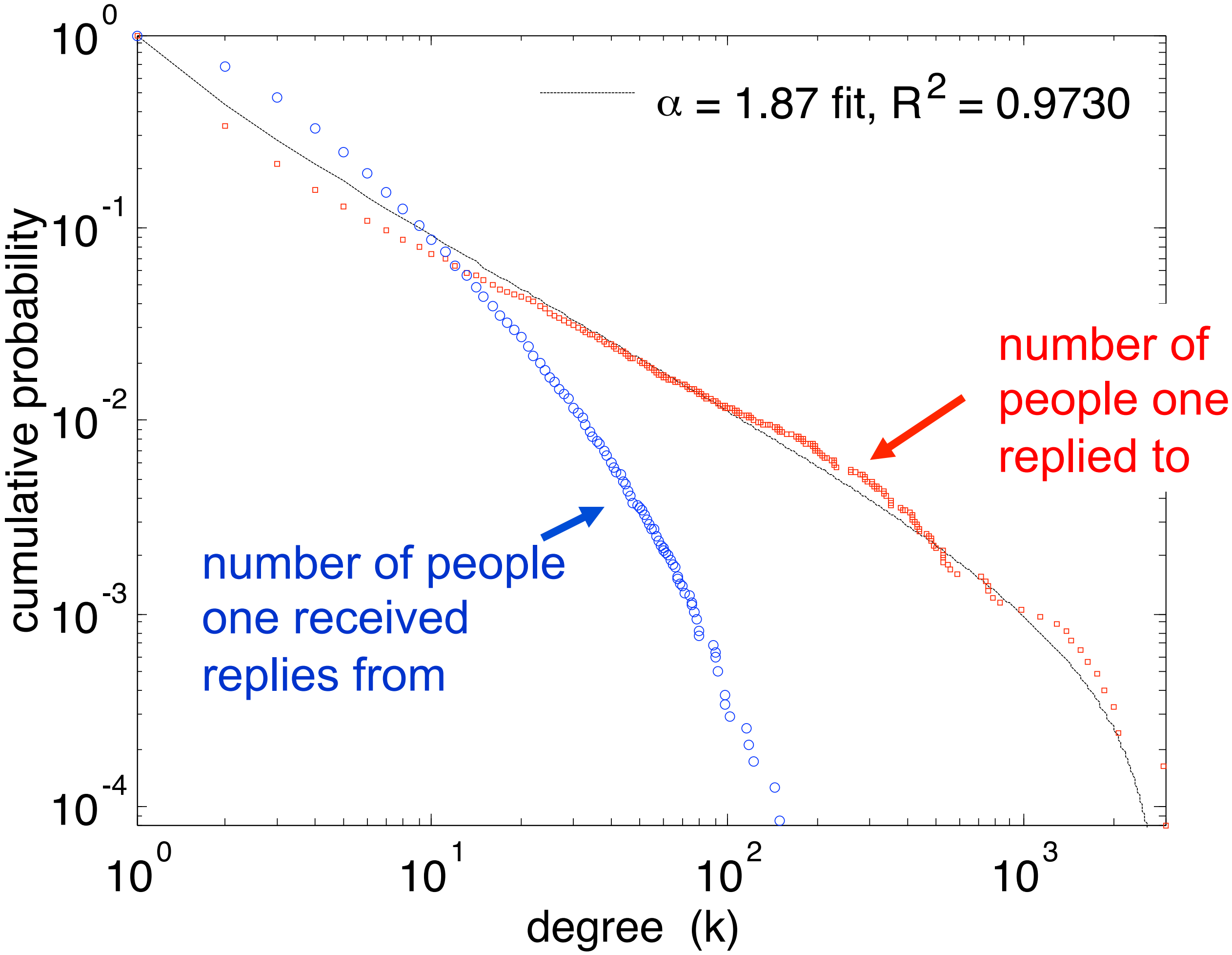
school of information

Online Question & Answer Forums



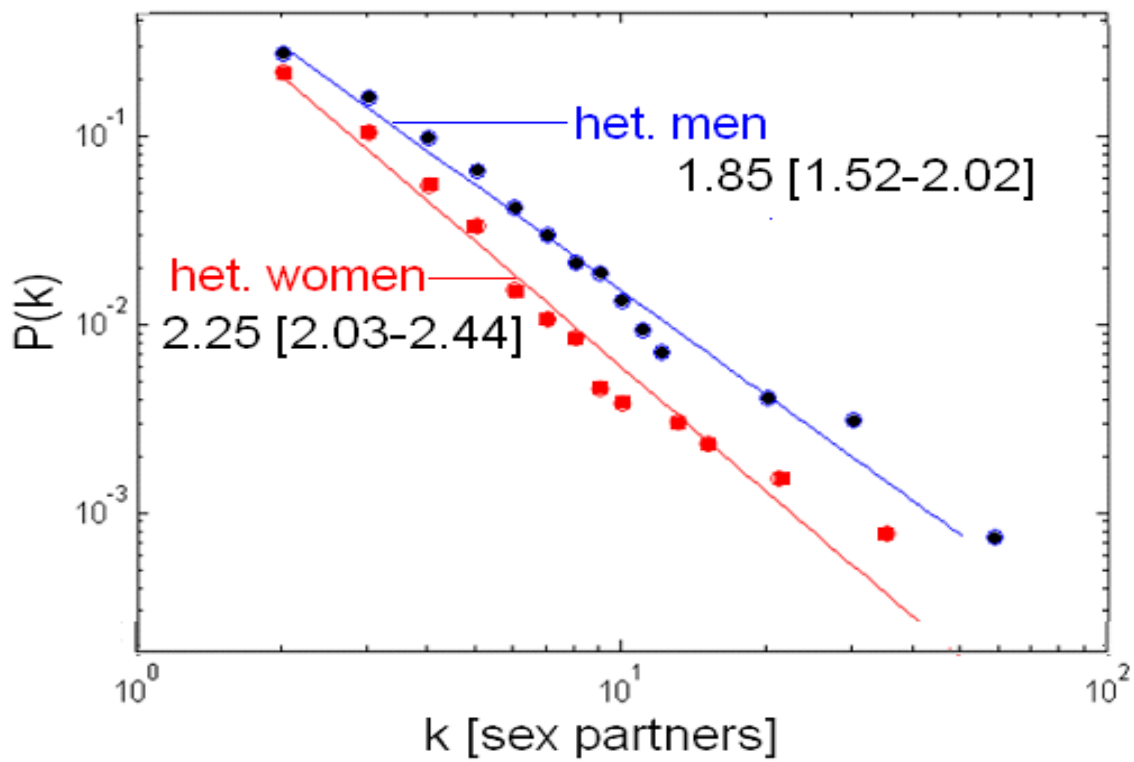
Uneven participation



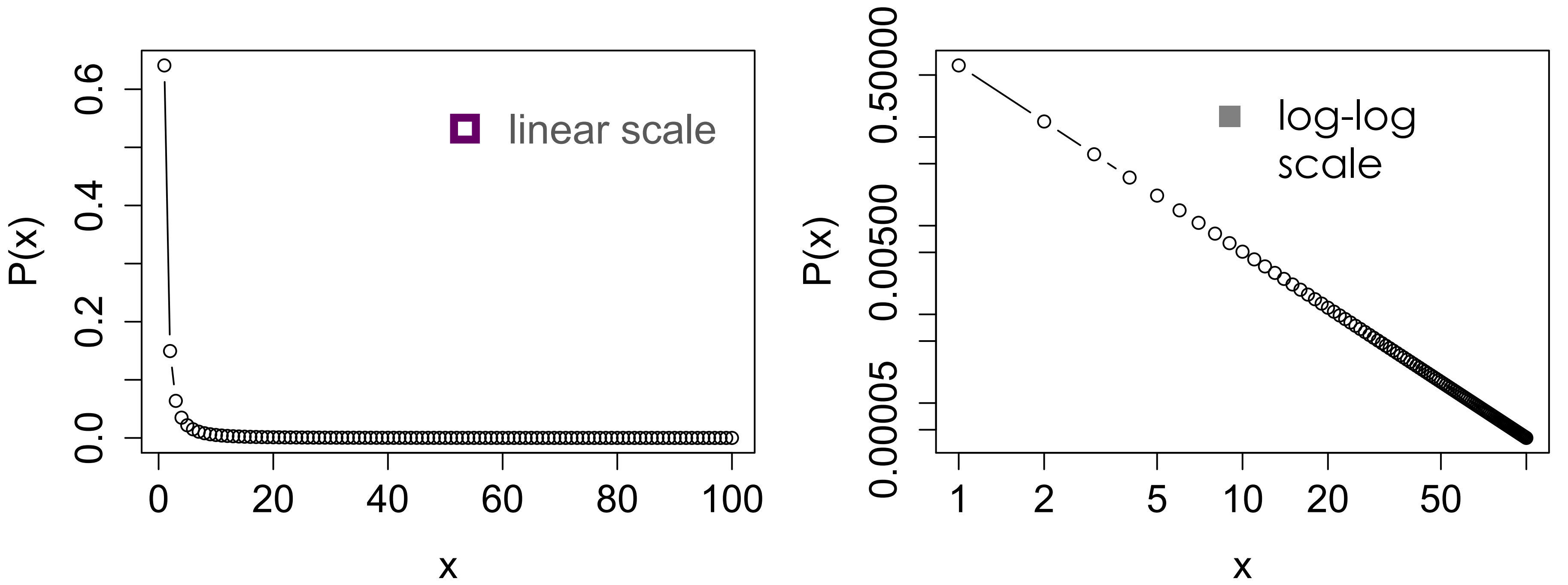
- 'answer people' may reply to thousands of others
- 'question people' are also uneven in the number of repliers to their posts, but to a lesser extent

Real-world degree distributions

- Sexual networks
- Great variation in contact numbers

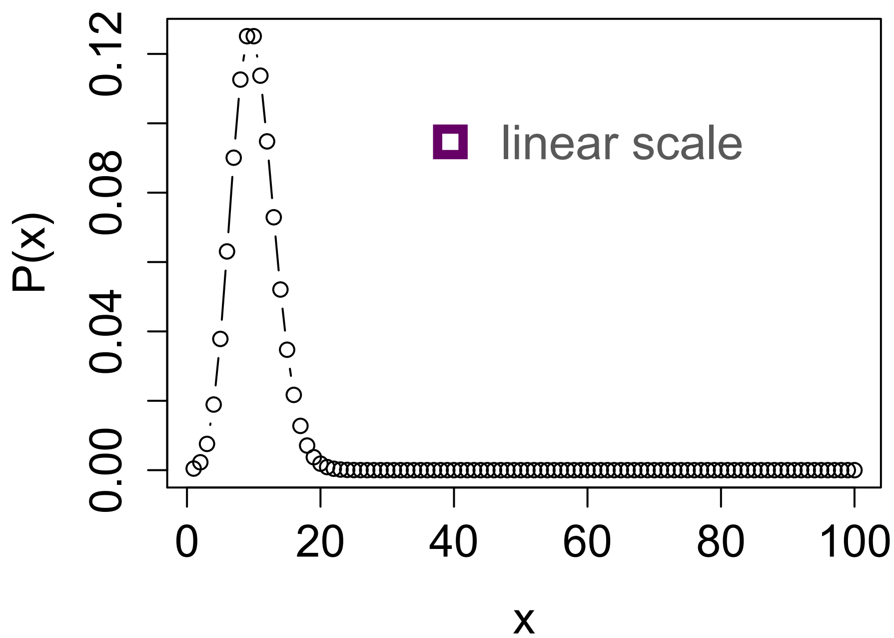


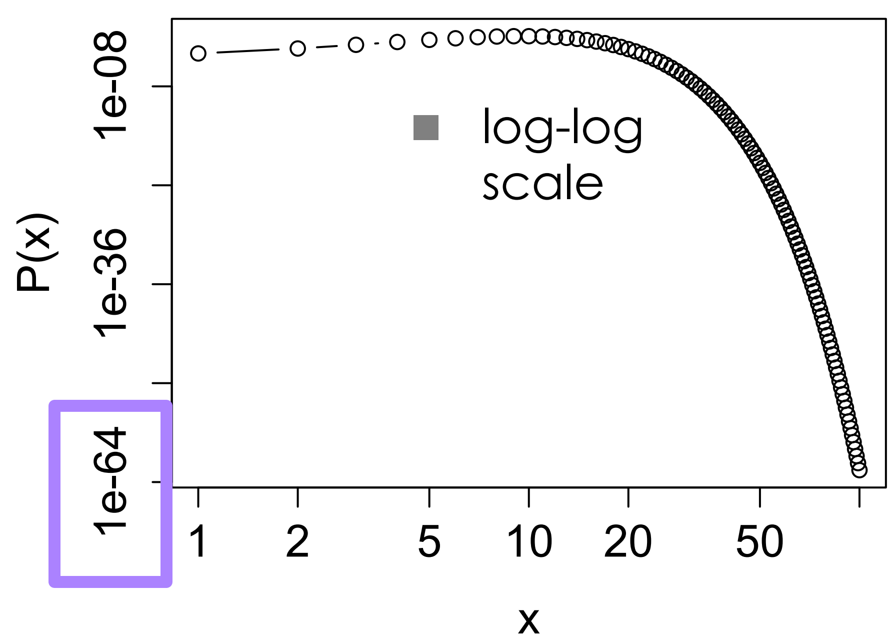
Power-law distribution



- high skew (asymmetry)
- straight line on a log-log plot

Poisson distribution





- little skew (asymmetry)
- curved on a log-log plot

Power law distribution

- Straight line on a log-log plot

$$\ln(p(k)) = c - \alpha \ln(k)$$

- Exponentiate both sides to get that *p(k)*, the probability of observing an node of degree 'k' is given by

$$p(k) = Ck^{-\alpha}$$

normalization constant (probabilities over all *k* must sum to 1)

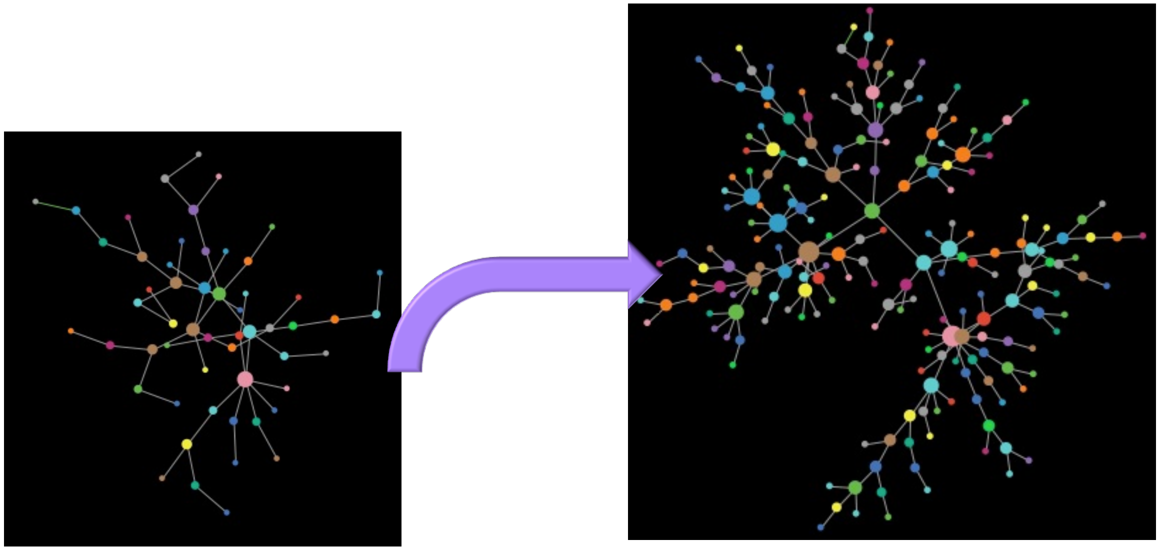
power law exponent α

Quiz Q:

- As the exponent α increases, the downward slope of the line on a log-log plot
  - stays the same
  - becomes milder
  - becomes steeper

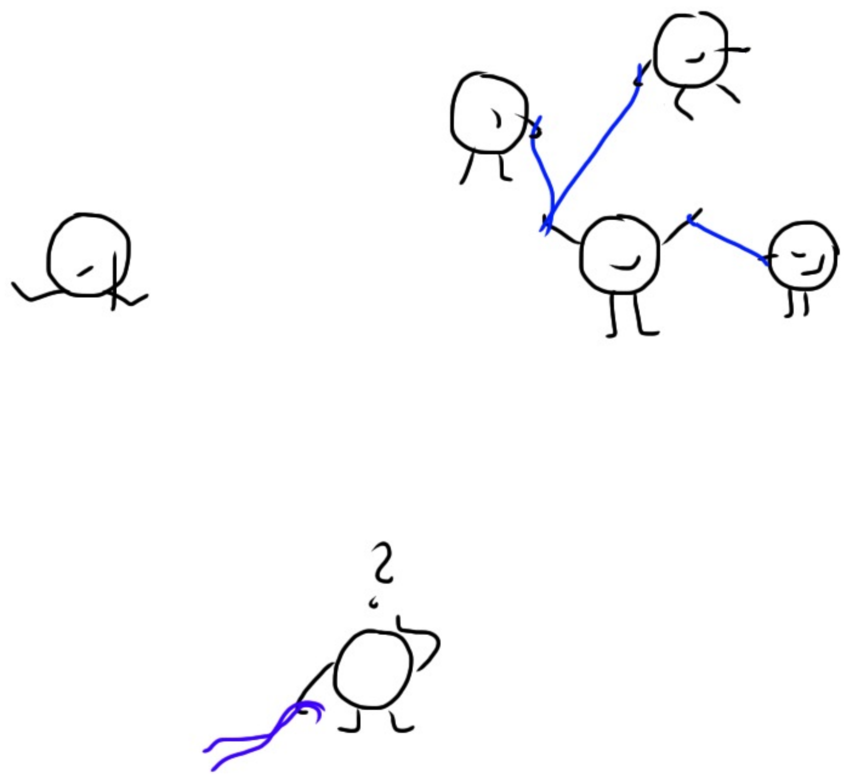
2 ingredients in generating power-law networks

- nodes appear over time (growth)



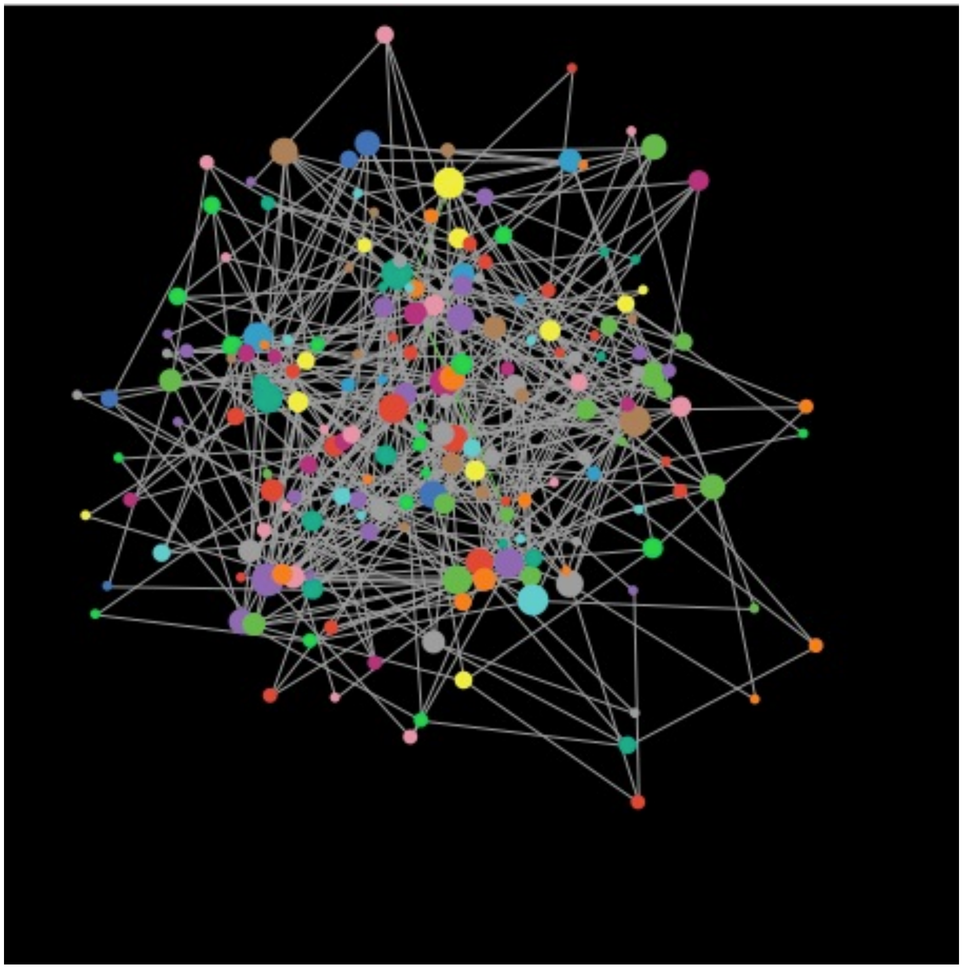
2 ingredients in generating power-law networks

- nodes prefer to attach to nodes with many connections (preferential attachment, cumulative advantage)



Ingredient # 1: growth over time

- nodes appear one by one, each selecting ***m*** other nodes at random to connect to



m = 2

random network growth

- one node is born at each time tick
- at time ***t*** there are ***t*** nodes
- change in degree k_i of node *i* (born at time i, with $0 < i < t$)

$$\frac{dk_i(t)}{dt} = \frac{m}{t}$$

there are ***m*** new edges being added per unit time (with 1 new node)

the ***m*** edges are being distributed among ***t*** nodes

a node in a randomly grown network

- how many new edges does a node accumulate since it's birth at time *i* until time *t*?
- integrate from *i* to *t*

$$\frac{dk_i(t)}{dt} = \frac{m}{t}$$

to get

$$k_i(t) = m + m\log\left(\frac{t}{i}\right)$$

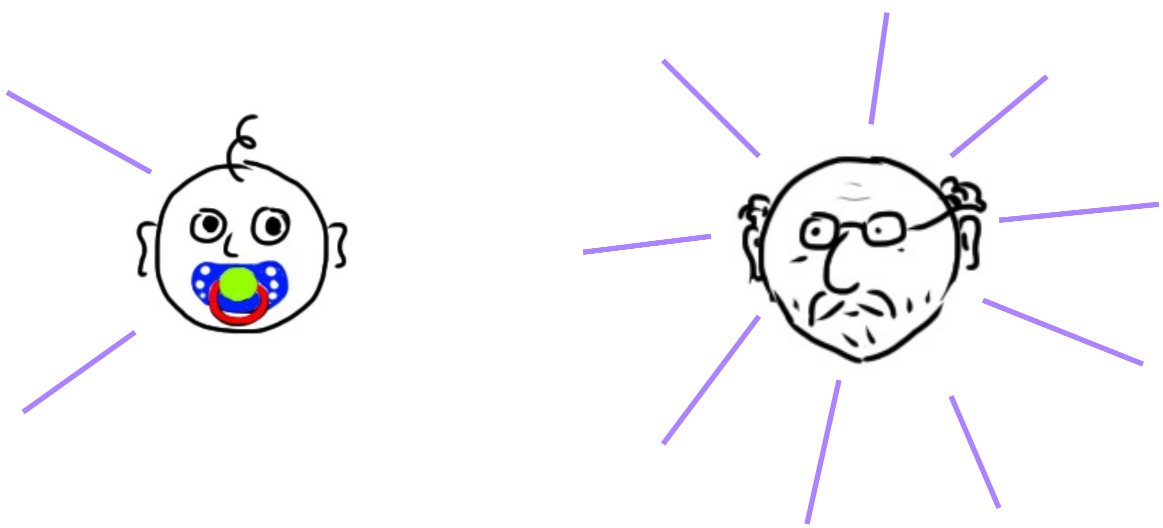
born with ***m*** edges

age and degree

on average $k_i(t) > k_j(t)$

if $i < j$

i.e. older nodes on average have more edges



Quiz Q:

- How could one make the growth model more realistic for social networks?
  - old nodes die
  - some nodes are more sociable
  - friendships vane over time
  - all of the above

growing random networks

Let $\tau(100)$ be the time at which node with degree e.g. 100 is born. The the fraction of nodes that have degree <= 100 is $(t - \tau)/t$

$$k_\tau(t) = m + m\log(\frac{t}{\tau})$$

random growth: degree distribution

- continuing…

$$\log\left(\frac{t}{\tau}\right) = \frac{k-m}{m}$$

$$\frac{\tau}{t} = e^{-\frac{k-m}{m}}$$

exponential distribution in degree

The probability that a node has degree *k* or less is $1-\tau/t$

$$P(k < k') = 1 - e^{-\frac{k'-m}{m}}$$

Quiz Q:

- The degree distribution for a growth model where new nodes attach to old nodes at random will be
  - a curved line on a log-log plot
  - a straight line on a log-log plot

2nd ingredient: preferential attachment

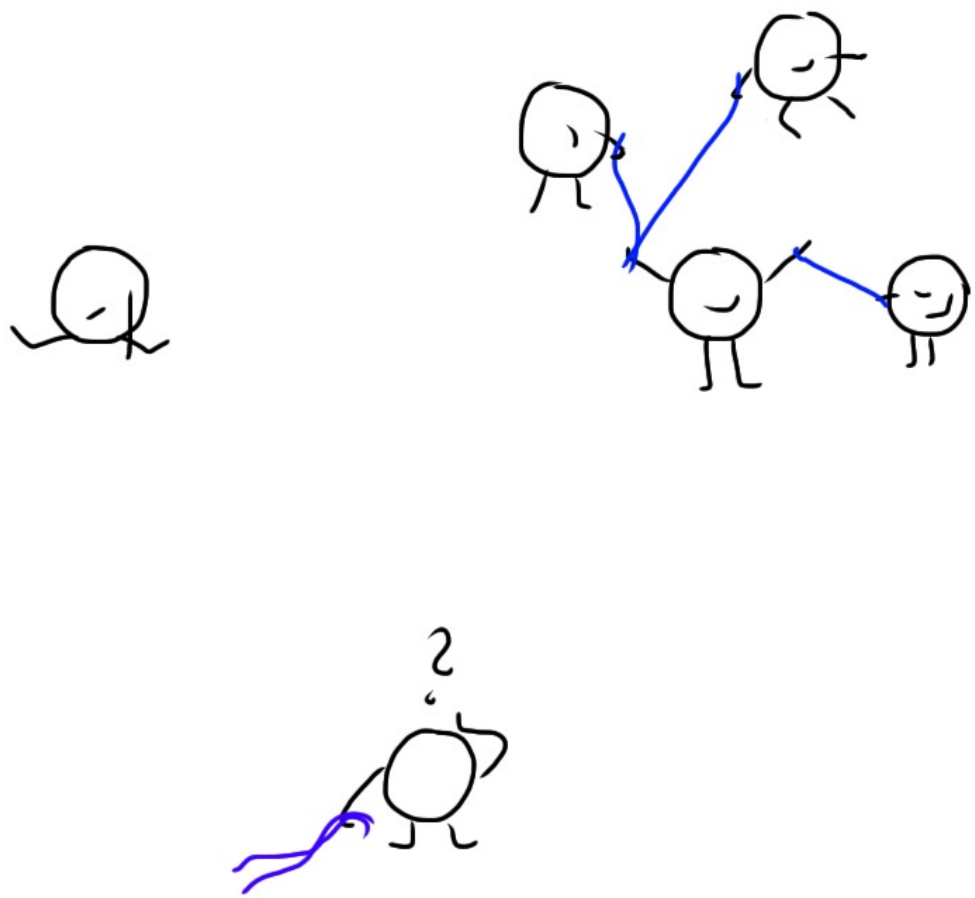
- Preferential attachment:
  - new nodes prefer to attach to well-connected nodes over less-well connected nodes
- Process also known as
  - cumulative advantage
  - rich-get-richer
  - Matthew effect

Price's preferential attachment model for citation networks

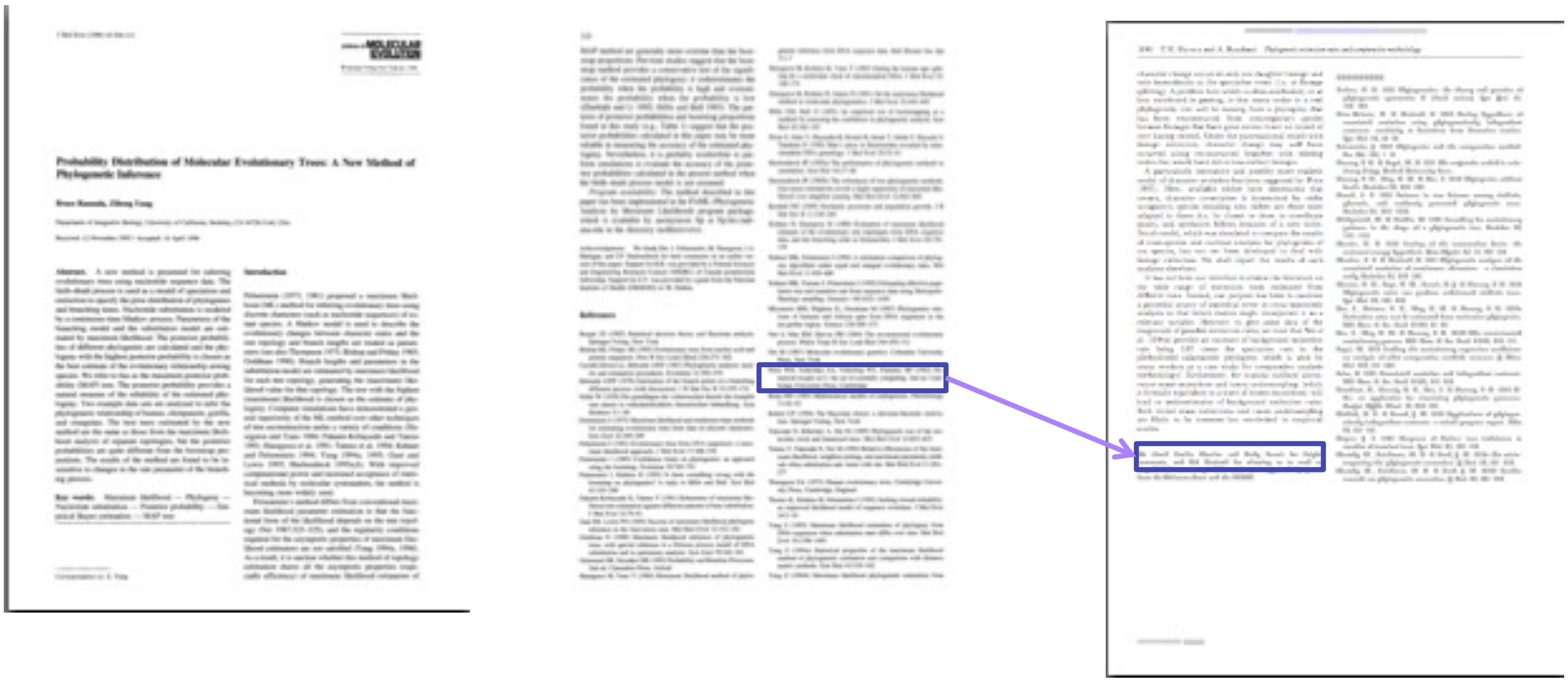
- [Price 65]
  - each new paper is generated with m citations (mean)
  - new papers cite previous papers with probability proportional to their indegree (citations)
  - what about papers without any citations?
    - each paper is considered to have a "default" citation
    - probability of citing a paper with degree k, proportional to k+1

- Power law with exponent $\alpha = 2+1/m$

Preferential attachment



Cumulative advantage: how?



- copying mechanism
- visibility

Barabasi-Albert model

- First used to describe skewed degree distribution of the World Wide Web

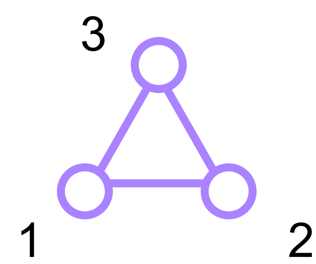
- Each node connects to other nodes with probability proportional to their degree
  - the process starts with some initial subgraph
  - each new node comes in with *m* edges
  - probability of connecting to node *i*

$$\Pi(i) = m \frac{k_i}{\sum_j k_j}$$

  - Results in power-law with exponent α = 3

Basic BA-model

- Very simple algorithm to implement
  - start with an initial set of m_0 fully connected nodes
    - e.g. $m_0 = 3$



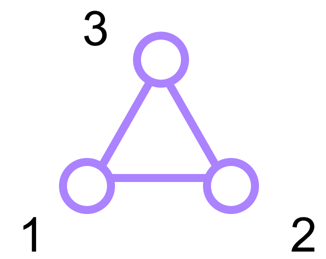
  - now add new vertices one by one, each one with exactly m edges
  - each new edge connects to an existing vertex in proportion to the number of edges that vertex already has → ***preferential attachment***
  - easiest if you keep track of edge endpoints in one large array and select an element from this array at random
    - the probability of selecting any one vertex will be proportional to the number of times it appears in the array – which corresponds to its degree

1 1 2 2 2 3 3 4 5 6 6 7 8 ....

generating BA graphs – cont'd

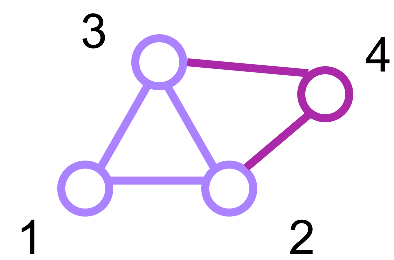
- To start, each vertex has an equal number of edges (2)
  - the probability of choosing any vertex is 1/3

1 1 2 2 3 3

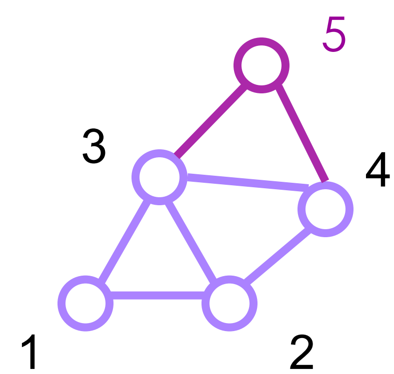


- We add a new vertex, and it will have m edges, here take m=2
  - draw 2 random elements from the array – suppose they are 2 and 3

1 1 2 2 2 3 3 3 4 4



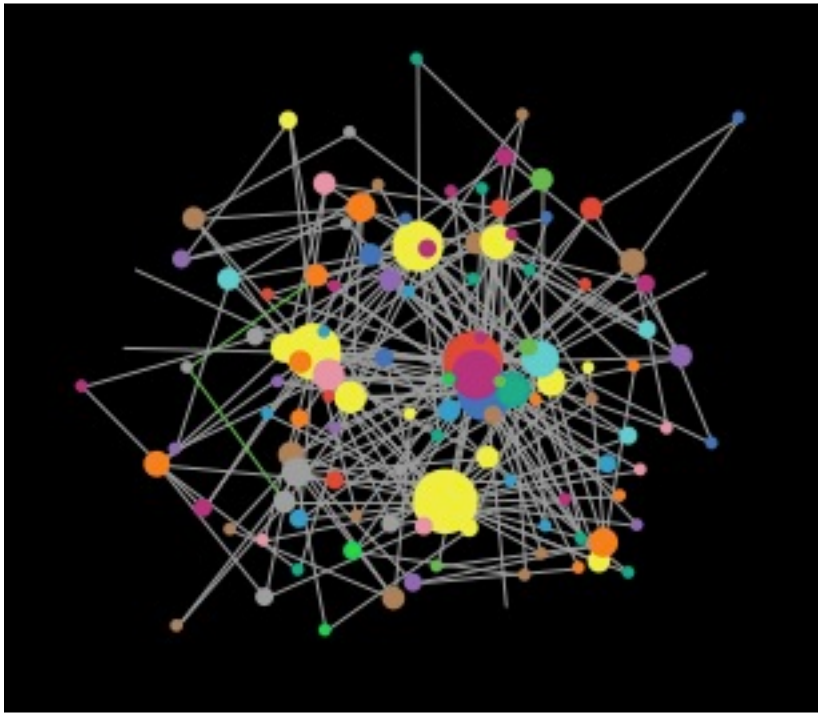
- Now the probabilities of selecting 1,2,3,or 4 are
  1/5, 3/10, 3/10, 1/5



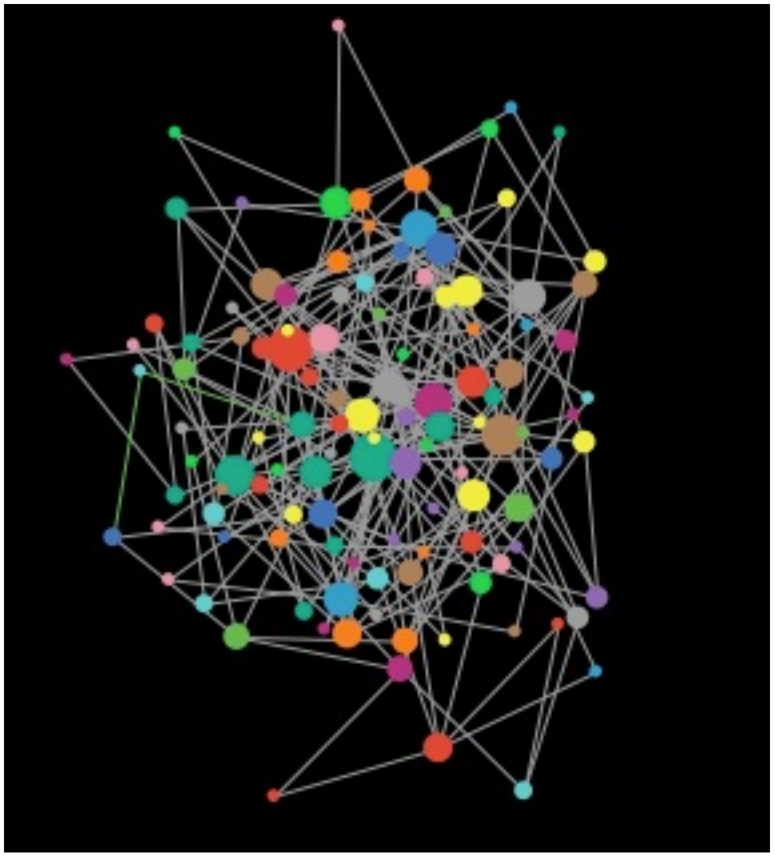
- Add a new vertex, draw a vertex for it to connect from the array
  - etc.

1 1 2 2 2 3 3 3 3 4 4 4 5 5

after a while...

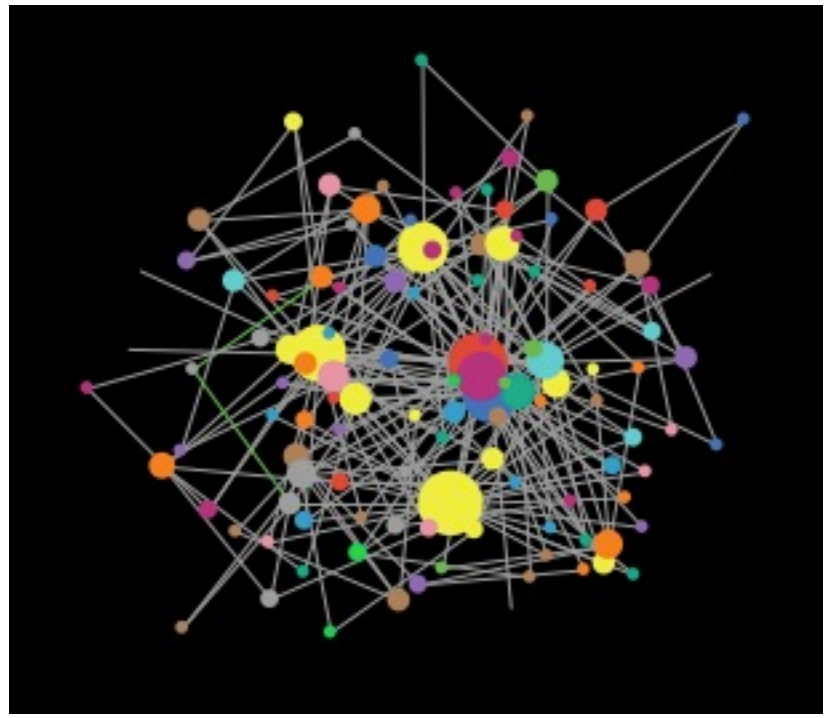


contrasting with random (non-preferential) growth



random

m = 2



preferential

mean field approximation

- probability that node ***i*** acquires a new link at time ***t***

$$\frac{dk_i(t)}{dt} = m\frac{k_i}{2tm} = \frac{k_i}{2t} \quad \text{with} \quad k_i(i) = m$$

$$k_i(t) = m(\frac{t}{i})^{1/2}$$

BA model degree distribution

- time of birth of node of degree k′: τ

$$\frac{\tau}{t} = \left(\frac{m}{k'}\right)^2$$

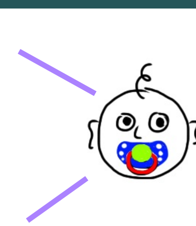
$$P(k < k') = 1 - \frac{m^2}{k'^2}$$

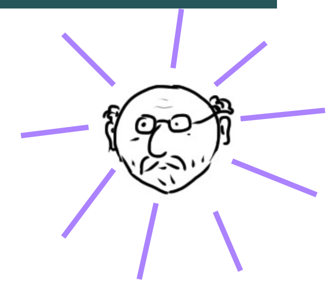
$$p(k) = \frac{2m^2}{k^3}$$

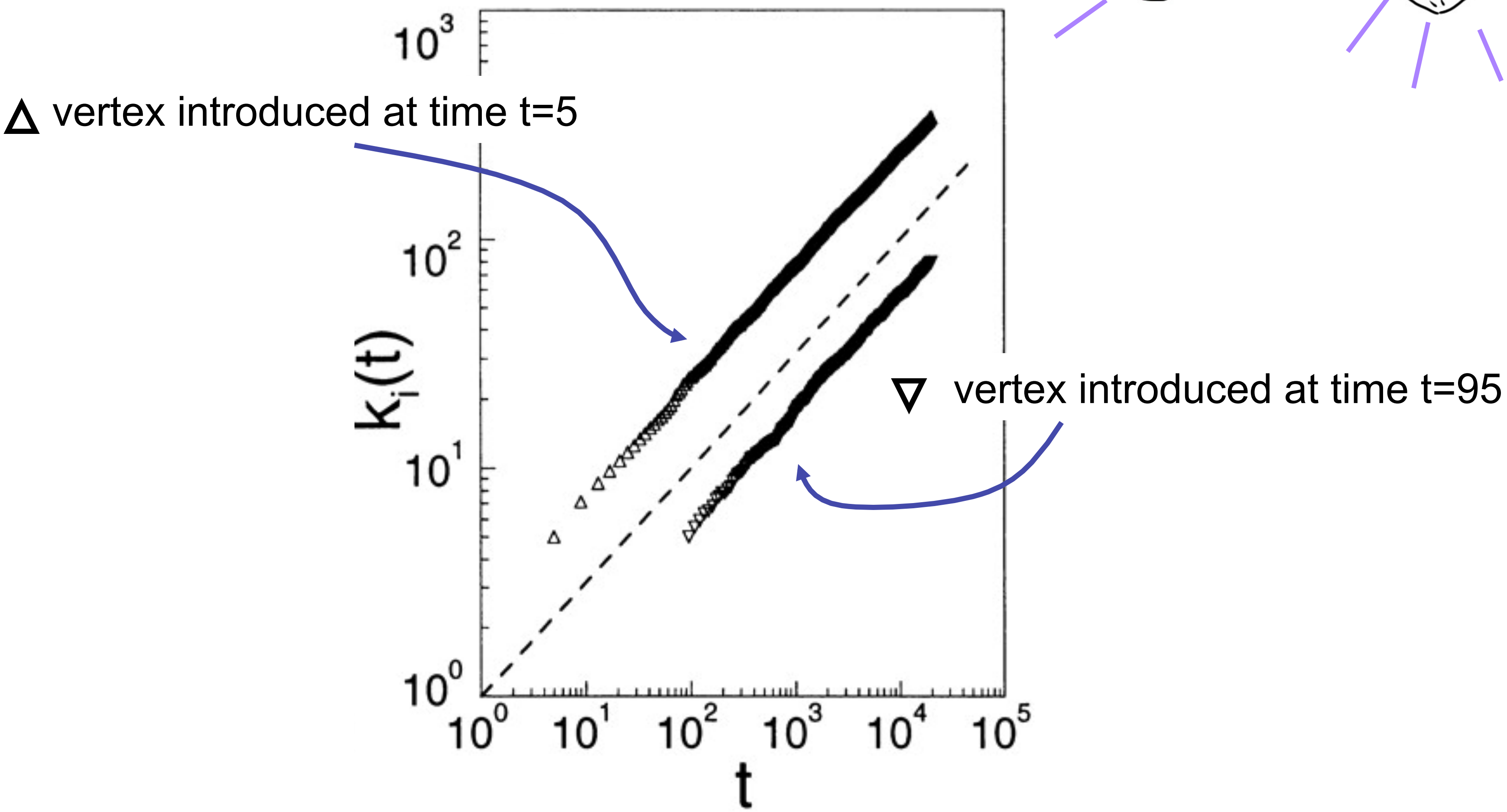
Properties of the BA graph

- The distribution is scale free with exponent $\alpha = 3$
  $P(k) = 2\ m^2/k^3$
- The graph is connected
  - Every new vertex is born with a link or several links (depending on whether $m = 1$ or $m > 1$)
  - It then connects to an 'older' vertex, which itself connected to another vertex when it was introduced
  - And we started from a connected core
- The older are richer
  - Nodes accumulate links as time goes on, which gives older nodes an advantage since newer nodes are going to attach preferentially – and older nodes have a higher degree to tempt them with than some new kid on the block

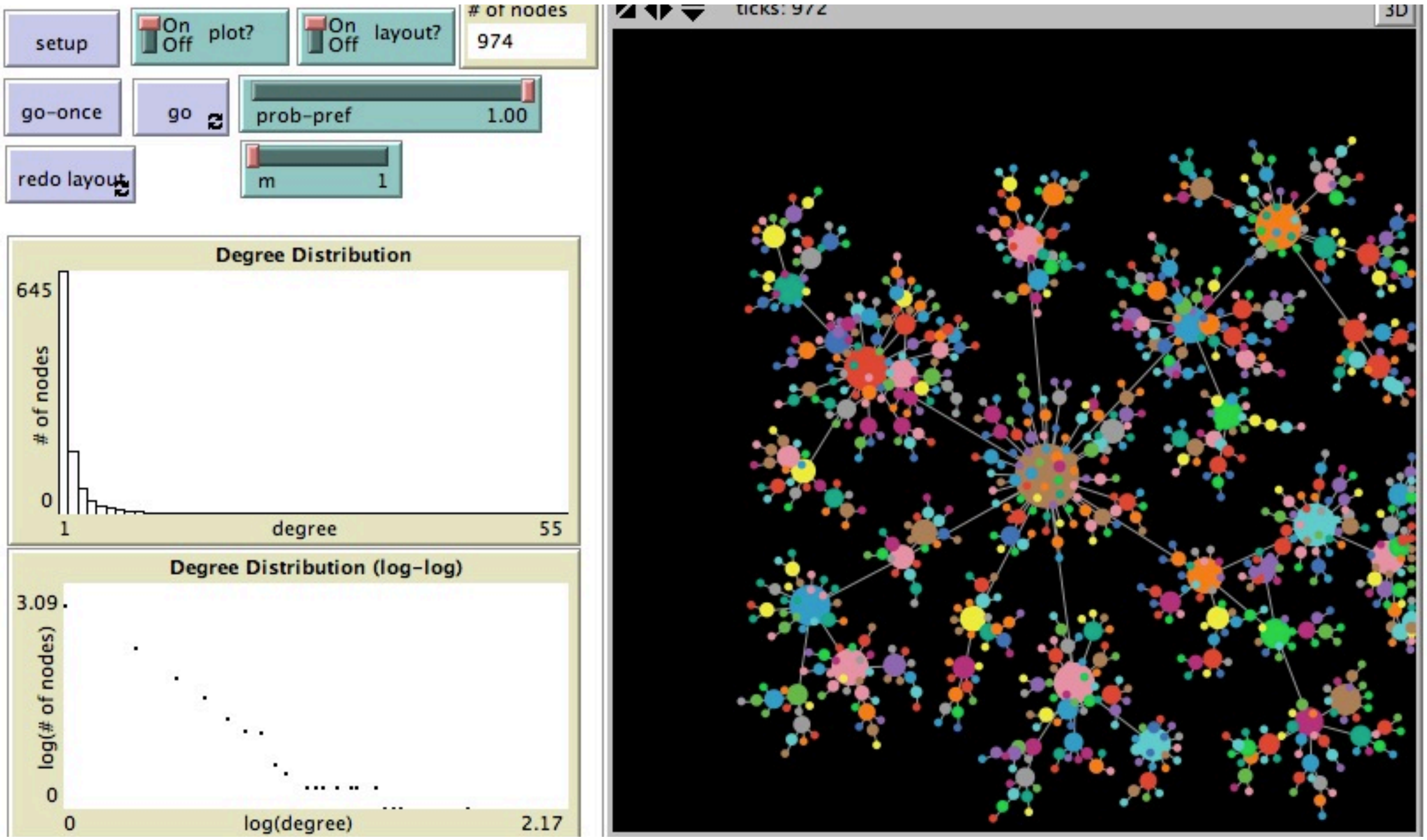
Young vs. old in BA model







try it yourself



http://www.ladamic.com/netlearn/NetLogo501/RAndPrefAttachment.html

Quiz Q:

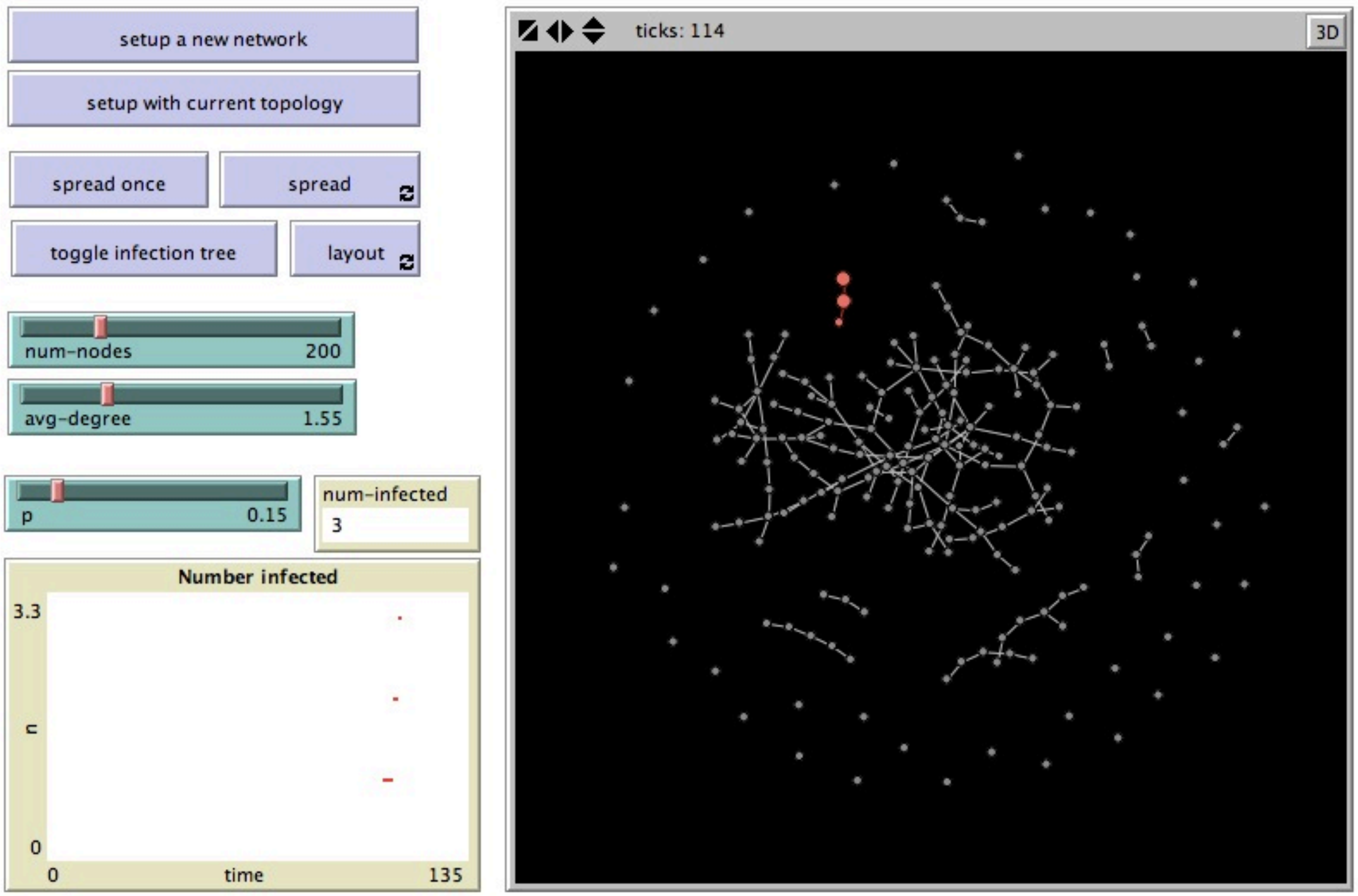
- Relative to the random growth model, the degree distribution in the preferential attachment model
  - resembles a power-law distribution less
  - resembles a power-law distribution more

Summary: growth models

- Most networks aren't 'born', they are made.
- Nodes being added over time means that older nodes can have more time to accumulate edges
- Preference for attaching to 'popular' nodes further skews the degree distribution toward a power-law

Assignment: implications for diffusion

- How does the size of the giant component influence diffusion?



Assignment: implications for diffusion

- How do growth and preferential attachment influence diffusion?