ÖDEV V Cevapları

① X RD çekilen 3 yumurta arasında bozuk olan yumurta sayısını versin

$$pf_X(r) = \frac{\binom{4}{r}\binom{12-4}{3-r}}{\binom{12}{3}}, \quad r = 0,1,2,3$$

$$pf_X(r) = \begin{cases} \frac{56}{220}, & r=0 \\ \frac{112}{220}, & r=1 \\ \frac{48}{220}, & r=2 \\ \frac{4}{220}, & r=3 \\ 0, & \text{diğer} \end{cases}$$

$$F_X(r) = \begin{cases} \frac{56}{220}, & r \le 0 \\ \frac{168}{220}, & 0 < r \le 1 \\ \frac{216}{220}, & 1 < r \le 2 \\ 1, & 2 < r \end{cases}$$

$F(1) = P(X \leq 1) = \frac{168}{220}$

$E(X) = \sum_{r_i} r_i\, pf_X(r_i)$

$= 0 \cdot \frac{56}{220} + 1 \cdot \frac{112}{220} + 2 \cdot \frac{48}{220} + 3 \cdot \frac{6}{220}$

$= 1$

$P(X \geq k) < 0{,}5$ tümleyenden gösterse k

$1 - P(X < k) \geq 1 - 0{,}5 \Rightarrow k = 1$

Bu durum %50 olasılıkla bozuk yumu tuların sayısının 1'den küçük olma ola lığını gösterir.

2) a) $P(0 < X < 1) = \int_0^1 pf_X(r)\,dr = \int_0^1 r\,dr = \frac{1}{2}$

b) $P(X > 1{,}5) = \int_{1{,}5}^{2} pf_X(r)\,dr = \int_{1{,}5}^{2} (2-r)\,dr = \frac{0{,}}{2}$

c) $E(X) = \int_0^1 r r\,dr + \int_1^2 r(2-r)\,dr = 1$

$$Var(X) = E(X^2) - (E(X))^2 \quad \text{où}$$

$$E(X^2) = \int_{\mathbb{R}} r^2 \, pf_X(r) \, dr$$

$$= \int_0^1 r r^2 dr + \int_1^2 (2-r) r^2 dr = 7/6$$

$$\Rightarrow Var(X) = 7/6 - 1 = 1/6$$

e) $$F_X(r) = \begin{cases} 0, & r \leq 0 \\ \int_0^r u\,du, & 0 < r < 1 \\ \int_0^1 u\,du + \int_1^r (2-u)\,du, & 1 \leq r < 2 \\ 1, & 2 \leq r \end{cases}$$

3) a) $$E(X) = \sum_{r_1} \sum_{r_2} r_1 \, pf_{(X,Y)}(r_1, r_2)$$

$$= \sum_{r_1=0}^{2} \sum_{r_2=0}^{3} r_1 \, pf_{(X,Y)}(r_1, r_2)$$

$$= \sum_{r_1=0}^{2} r_1 \Big(pf_{(X,Y)}(r_1, 0) + pf_{(X,Y)}(r_1, 1) + pf_{(X,Y)}(r_1, 2) + pf_{(X,Y)}(r_1, 3) \Big)$$

$$= pf_{(X,Y)}(1,0) + pf_{(X,Y)}(1,1) + pf_{(X,Y)}(1,2) + pf_{(X,Y)}(1,3)$$

$$+2\left(pf_{(X,Y)}(2,0) + pf_{(X,Y)}(2,1) + pf_{(X,Y)}(2,2) + pf_{(X,Y)}(2,3)\right)$$

$$= \frac{1}{30}\left(1+2+3+4\right) + \frac{2}{30}\left(2+3+4+5\right) = \frac{38}{30}$$

$$E(X^2) = \sum_{r_1=0}^{2} \sum_{r_2=0}^{3} r_1^2 \, pf_X(r_1, r_2) = \frac{22}{30}$$

$$\Rightarrow Var(X) = E(X^2) - \left(E(X)\right)^2 = \frac{22}{30} - \left(\frac{38}{30}\right)^2$$

b) $E(X|Y)$ degerini bulabilmek için önceli le koşullu olasılık fonk'unu bulmak gerekle

$$pf_{(X,Y)}(r_1|r_2) = \frac{pf_{(X,Y)}(r_1, r_2)}{pf_Y(r_2)}$$

$$= \frac{\frac{1}{30}(r_1+r_2)}{\sum_{r_1=0}^{2} \frac{1}{30}(r_1+r_2)} = \frac{\frac{1}{30}(r_1+r_2)}{\frac{1}{10}(1+r_2)}$$

$$= \frac{1}{3} \frac{(r_1+r_2)}{(r_2+1)}$$

$$E(X|Y) = \sum_{r_1=0}^{2} r_1 p f_{(X,Y)}(r_1|r_2)$$

$$= \sum_{r_1=0}^{3} r_1 \frac{1}{3} \frac{r_1+r_2}{1+r_2} = \frac{5+3r_2}{3(r_2+1)}$$

$$E(X|Y=2) = \frac{5+3r_2}{3(r_2+1)}\Bigg|_{r_2=2} = \frac{11}{9}$$

$$E(XY) = \sum_{r_1=0}^{2} \sum_{r_2=0}^{3} r_1 r_2 \frac{1}{30}(r_1+r_2)$$

$$= \sum_{r_1=0}^{2} \left[\frac{r_1}{30}(r_1+1) + \frac{2r_1}{30}(r_1+2) + \frac{3r_1}{30}(r_1+3 \right.$$

$$= \frac{1}{30}2 + \frac{2}{30}3 + \frac{3}{30}4 + \frac{2}{30}3 + \frac{4}{30}4 + \frac{6}{30}5$$

$$= \frac{1}{30}\left(2+6+12+6+16+30\right) = \frac{72}{30}$$

4) Olayları tanımlayalım

$A = \{$ Çekilen topun kırmızı olması $\}$

$B_1 = \{$ 1. kutudan çekilmesi $\}$

$B_2 = \{$ 2. " " $\}$

$B_3 = \{$ 3. " " $\}$

Toplam olasılık formülünden

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)$$

$$= \frac{1}{3}\frac{2}{10} + \frac{1}{3}\frac{4}{8} + \frac{1}{3}\frac{3}{10} = 1/3$$ bulunur.

Bayes teoreminin uygulamasıyla istenen olasılık

$$P(B_3|A) = \frac{P(B_3 \cap A)}{P(A)} = \frac{P(B_3)P(A|B_3)}{P(A)}$$

$$= \frac{\frac{1}{3}\frac{3}{10}}{1/3} = \frac{3}{10}$$

5) $S = \{1,2,3,4,5,6\}$

$$X: S \longrightarrow \mathbb{R}$$
$$s \longmapsto X(s) = 2s$$

$$Y: S \longrightarrow \mathbb{R}$$
$$s \longmapsto Y(s) = \begin{cases} 1, & s = 1,3,5 \\ 3, & s = 2,4,6 \end{cases}$$

(i) $$pf_X(r) = \begin{cases} \frac{1}{6}, & r = 2,4,6,8,10,12 \\ 0, & \text{diğer} \end{cases}$$

$$E(X) = \sum_{\substack{k=1 \\ r=2k}}^{6} r\, pf_X(r)$$

$$= 2\frac{1}{6} + 4\frac{1}{6} + 6\frac{1}{6} + 8\frac{1}{6} + 10\frac{1}{6} + 12\frac{1}{6}$$

$$= \frac{1}{6}42 = 7$$

$$E(X^2) = \sum_{\substack{k=1 \\ r=2k}}^{6} r^2 pf_X(r) = \frac{1}{6}(2^2+4^2+6^2+8^2+10^2+12^2)$$

$$= 364/6$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{364}{6} - 7^2$$

$$\sigma(X) = \sqrt{Var(X)} = \sqrt{\frac{364}{6} - 7^2}$$

(ii) $$pf_Y(r) = \begin{cases} \frac{1}{2}, & r = 1, 3 \\ 0, & \text{διαφ} \end{cases}$$

$$E(Y) = \sum_{\substack{k=1 \\ r=2k+1}}^{2} r\, pf_Y(r) = 1 \cdot \frac{1}{2} + 3\, \frac{1}{2} = 2$$

$$E(Y^2) = \sum_{\substack{k=1 \\ r=2k+1}}^{2} r^2\, pf_Y(r) = 1\frac{1}{2} + 3^2\frac{1}{2} = 5$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = 5 - 2^2 = 1$$

$$\sigma(Y) = \sqrt{Var(Y)} = 1$$

(iii) $$X+Y : S \longrightarrow \mathbb{R}$$
$$s \longmapsto (X+Y)(s) = X(s) + Y(s)$$
$$R(X+Y) = \{3, 7, 11, 15\}$$

$$(X+Y)(1) = X(1) + Y(1) = 2 + 1 = 3$$
$$(X+Y)(2) = X(2) + Y(2) = 4 + 3 = 7$$
$$(X+Y)(3) = X(3) + Y(3) = 6 + 1 = 7$$
$$(X+Y)(4) = X(4) + Y(4) = 8 + 3 = 11$$
$$(X+Y)(5) = X(5) + Y(5) = 10 + 1 = 11$$
$$(X+Y)(6) = X(6) + Y(6) = 12 + 3 = 15$$

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$$pf_{(X+Y)}(r) = \begin{cases} 1/6 & , r=3 \\ 2/6 & , r=7 \\ 2/6 & , r=11 \\ 1/6 & , r=15 \\ 0 & , \text{diğer} \end{cases}$$

$$E(X+Y) = 3\cdot\frac{1}{6} + 7\frac{2}{6} + 11\cdot\frac{2}{6} + 15\frac{1}{6} = \frac{54}{6}$$

$$E((X+Y)^2) = 3^2\frac{1}{6} + 7^2\frac{2}{6} + 11^2\frac{2}{6} + 15^2\frac{1}{6} = \frac{574}{6}$$

$$Var(X+Y) = E((X+Y)^2) - \left(E(X+Y)\right)^2$$

$$= \frac{574}{6} - \left(\frac{54}{6}\right)^2$$

$$\sigma(X+Y) = \sqrt{Var(X+Y)}$$

Dikkat edilirse $E(X+Y) = 9 = 7+2 = E(X)+E(Y)$ olmasına rağmen $Var(X+Y) \neq Var(X)+Var(Y)$ olur.

(iv) $X \cdot Y : S \longrightarrow \mathbb{R}$

$s \longmapsto (X\cdot Y)(s) = X(s)\,Y(s)$

$R(XY) = \{2, 6, 10, 12, 24, 36\}$

$(XY)(1) = X(1)Y(1) = 2$

$(XY)(2) = X(2)\,Y(2) = 12$

$(XY)(3) = X(3)Y(3) = 6$

$(XY)(4) = X(4)Y(4) = 24$

$(XY)(5) = X(5)Y(5) = 10$

$(XY)(6) = X(6)Y(6) = 36$

$$pf_{XY}(r) = \begin{cases} \frac{1}{6} & , r = 2, 6, 10, 12, 24, 36 \\ 0 & , \text{diger} \end{cases}$$

$$E(XY) = \frac{1}{6}\left(2 + 6 + 10 + 12 + 24 + 36\right) = \frac{90}{6} = 15$$

$$E((XY)^2) = \frac{1}{6}\left(2^2 + 6^2 + 10^2 + 12^2 + 24^2 + 36^2\right) = \frac{2156}{6}$$

$$Var(XY) = E((XY)^2) - (E(XY))^2$$

$$\sigma(XY) = \sqrt{Var(XY)}$$

b) $$pf_X(r_1) = \begin{cases} 1/2 & , r_1 = 1 \\ 1/2 & , r_1 = 3 \\ 0 & , \text{diger} \end{cases}$$

$$pf_Y(r_2) = \begin{cases} 3/10 & , r_2 = 2 \\ 4/10 & , r_2 = 3 \\ 3/10 & , r_2 = 4 \\ 0 & , \text{diger} \end{cases}$$

$$E(X) = 1 \cdot \frac{1}{2} + 3\,\frac{1}{2} = 2$$

$$E(Y) = 2 \cdot \frac{3}{10} + 3\,\frac{4}{10} + 4 \cdot \frac{3}{10} = \frac{30}{10} = 3$$

$$E(XY) = \sum_{r_1} \sum_{r_2} r_1 r_2\, pf_{\vec{X}}(r_1, r_2)$$

$$= \sum_{r_1} r_1 \Big[2\, pf_{\vec{X}}(r_1, 2) + 3\, pf_{\vec{X}}(r_1, 3) + 4\, pf_{\vec{X}}(r_1, 4)$$

$$= 1.2\, pf_{\vec{X}}(1,2) + 1.3\, pf_{\vec{X}}(1,3) + 1.4\, pf_{\vec{X}}(1,4)$$

$$+ 3.2\, pf_{\vec{X}}(3,2) + 3.3\, pf_{\vec{X}}(3,3) + 3.4\, pf_{\vec{X}}(3,4)$$

$$= 2 \cdot \frac{2}{10} + 3\frac{1}{10} + 4\frac{2}{10} + 6\frac{1}{10} + 9\frac{3}{10} + 12\frac{1}{10}$$

$$= \frac{1}{10}(4 + 3 + 8 + 6 + 27 + 12) = 6$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$
$$= 6 - 2.3 = 0$$

(ii) $E(X^2) = 1^2 \frac{1}{2} + 3^2 \frac{1}{2} = \frac{10}{2} = 5$

$$Var(X) = E(X^2) - (E(X))^2 = 5 - 2^2 = 1$$

$$\sigma(X) = \sqrt{Var(X)} = 1$$

$$E(Y^2) = 2^2 \frac{3}{10} + 3^2 \frac{4}{10} + 4^2 \frac{3}{10} = 9{,}6$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = 9{,}6 - 3^2 = 0{,}6$$

$$\sigma(Y) = \sqrt{Var(Y)} = \sqrt{0{,}6} = 0{,}77$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma(X)\sigma(Y)} = \frac{0}{96.\ 0{,}77} = 0.$$

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$P(X=1, Y=3) = \frac{1}{10} \neq \frac{1}{2}\cdot\frac{3}{10} = P(X=1)\cdot P(Y=3)$

olduğundan bağımsız değildirler.

7) $$pf_X(r) = \begin{cases} \frac{1}{8} & r=0 \\ \frac{3}{8} & r=1 \\ \frac{3}{8} & r=2 \\ \frac{1}{8} & r=3 \\ 0 & \text{diğer} \end{cases}$$

$$m_X(r) = \sum_{r=0}^{3} e^{tr}\, pf_X(r)$$

$$= \frac{1}{8}(1+3e^{t}+3e^{2t}+e^{3t})$$

$$= \frac{1}{8}(e^{t}+1)^3$$

8) $P(X>1000) \leq \frac{E(X)}{1000} = \frac{500}{1000} \leq \frac{1}{2}$

9) $P(|X-m| \leq k\,\sigma(X)) \geq 1-\frac{1}{k^2}$

$\Rightarrow m-k\sigma(X) \leq X \leq m+k\sigma(X)$

$\Rightarrow m-k\sigma(X) = -3$ ve $m+k\sigma(X) = 3$

$\boxed{k=3}$ bulunur.

$$E(Y) = E(X^3 + 3^X)$$

$$= \sum_{r=1}^{\infty} (r^3 + 3^r)\, p q^{r-1}$$

$$= \sum_{r=1}^{\infty} (r^3 + 3^r) \frac{3}{4} \left(\frac{1}{4}\right)^{r-1}$$

$$= 3 \sum_{r=1}^{\infty} (r^3 + 3^r) \frac{1}{4^r}$$

$$= 3 \underbrace{\sum_{r=1}^{\infty} \frac{r^3}{4^r}}_{(*)} + 3 \underbrace{\sum_{r=1}^{\infty} \left(\frac{3}{4}\right)^r}_{\text{geo seri yakınsa}}$$

$\sum_{r=0}^{\infty} r^3 x^r$ serisi için $\sum_{r=0}^{\infty} x^r$ serisini ele

alalım. $\sum_{r=0}^{\infty} x^r = \frac{1}{1-x}$, $|x| < 1$

$$1 + x + x^2 + \cdots = \frac{1}{1-x}, \quad |x| < 1$$

$$\frac{d}{dx}(1 + x + x^2 + \cdots) = \frac{d}{dx}\left(\frac{1}{1-x}\right), \quad |x| < 1$$

$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}, \quad |x| < 1$$

$$x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2}, \quad |x| < 1$$

$$\Rightarrow \frac{d}{dx}\left(x+2x^2+3x^3+\cdots\right)=\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right), |x|<$$

$$\Rightarrow 1+2^2x+3^2x^2+\cdots=\frac{(1-x)^2+x\,2(1-x)}{(1-x)^4}, |x|<$$

$$\Rightarrow 1+2^2x+3^2x^2+\cdots=\frac{1+x}{(1-x)^3}, \quad |x|<1$$

$$\Rightarrow x+2^2x^2+3^2x^3+\cdots=\frac{x(1+x)}{(1-x)^3}, \quad |x|<$$

$$\Rightarrow \frac{d}{dx}\left(x+2^2x^2+3^2x^3+\cdots\right)=\frac{d}{dx}\left(\frac{x(1+x)}{(1-x)^3}\right)$$

$$|x|<1$$

$$\Rightarrow 1+2^3x+3^3x^2+\cdots=\frac{(1+x+x)(1-x)^3+x(1+x)3(1-x)^2}{(1-x)^6}$$

$$=\frac{(1+2x)(1-x)+3x(1+x)}{(1-x)^4}, \quad |x|<1$$

$$\Rightarrow x+2^3x^2+3^3x^3+\cdots=x\,\frac{1-x+2x-2x^2+3x+3x}{(1-x)^4}$$

$$=x\,\frac{1+4x+x^2}{(1-x)^4}, \quad |x|<$$

$$\Rightarrow \sum_{r=0}^{\infty} r^3x^r = x\,\frac{(x^2+4x+1)}{(1-x)^4}, \quad |x|<$$

$$\Rightarrow (*) \quad \sum_{r=1}^{\infty} r^3 \frac{1}{4^r} = \frac{\frac{1}{4}\left(\frac{1}{4^2} + 4\cdot\frac{1}{4} + 1\right)}{\left(1-\frac{1}{4}\right)^4}$$

$$= \frac{\frac{1}{4}\left(2 + \frac{1}{16}\right)}{\frac{3^4}{4^4}} = \frac{4^4 \; 33}{\cancel{4\cdot 16}\cdot 3^4}$$

$$= \frac{4\cdot 33}{3^4}$$

$$\Rightarrow E(Y) = 3\cdot\frac{4\cdot 33}{3^4} + 3\cdot\frac{3}{4}\,\frac{1}{1-\frac{3}{4}}$$

$$= \frac{4\cdot 11}{9} + 9 = \frac{125}{9}$$