Fourier Analysis: Part II

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Fourier Series (Recap)

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Fourier Series with any period of p=2L (Recap)

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

Sturm-Liouville Problem (SLP)

$$\left[p(x)y'\right]' + [q(x) + \lambda r(x)]y = 0$$

on some interval $a \equiv b$, satisfying conditions of the form

1. $k_1 y + k_2 y' = 0 \quad$ at $x = a$
2. $l_1 y + l_2 y' = 0 \quad$ at $x = b$.

λ: parameter

k_1, k_2, l_1, $l;_2$: real constants

* one of the conditions must be satisfied by non-zero y.

The above equation and conditions are called *Sturm-Liouville Equation* and *Sturm-Liouville Problem*.

Orthogonal Functions

Orthogonal functions on $a \leqq x \leqq b$ with respect to the weight function $r(x) > 0$:

$$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x) dx = 0 \quad (m \neq n)$$

The norm $\|y_m\|$ of y_m is defined by

$$\|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x) y_m^2(x) dx}$$

The functions $y_1, y_2, \cdots$ are called orthonormal on $a \leqq x \leqq b$ if they are orthogonal on this interval and all have norm 1.

Orthogonal Series

Orthogonality will give us coefficient formulas for the desired generalised Fourier Series. Let y_0, y_1... be orthogonal wrt to a weight function $r(x)$. Then, a function:

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + ...$$

This is called an **orthogonal expansion** (series). The Fourier constants:

$$a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b r(x) f(x) y_m(x) dx \quad (n = 0, 1, \cdots)$$

See Fourier-Legendre and Fourier-Bessel series!

Fourier Integral

Can a similar approach to Fourier Series be applied for non-periodic functions?

- We know how to deal with periodic functions with a period of $2L$
- See what happens to FS if we let $L \to \infty$

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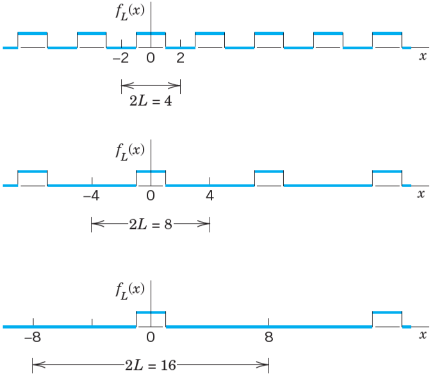


Fourier Integral

Any periodic function $f_L(x)$ of period $2L$ (with FS):

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_n x + b_n \sin \omega_n x), \quad \omega_n = \frac{n\pi}{L}$$

where (with dummy variable v)

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_L(v) dv, \quad a_n = \frac{1}{L}\int_{-L}^{L} f_L(v) \cos \omega_n v dv, \quad b_n = \frac{1}{L}\int_{-L}^{L} f_L(v) \sin \omega_n v dv$$

Let's set

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}.$$

and $1/L = \Delta\omega/\pi$.

Fourier Integral

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Bigg[(\cos *omega_n x)\,\Delta\omega \int_{-L}^{L} f_L(v)\cos\omega_n v dv + (\sin\omega_n x)\,\Delta\omega \int_{-L}^{L} f_L(v)\sin\omega_n v dv\Bigg].$$

$1/L \to 0$; therefore, the first term becomes 0. The sum term becomes integral:
$f(x) = \frac{1}{\pi}\int_0^\infty \left[\cos\omega x \int_{-\infty}^{\infty} f(v)\cos\omega v dv + \sin\omega x \int_{-\infty}^{\infty} f(v)\sin\omega v dv\right] d\omega$
The representation by a **Fourier integral**:

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x] d\omega$$

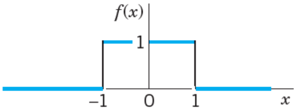
where

$$\boxed{A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v dv,} \quad \boxed{B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v dv.}$$

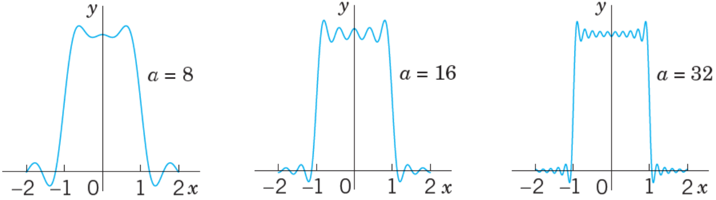
Fourier Integral (Example)

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$



Soln: $$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos \omega x \sin \omega}{\omega} d\omega}$$



Fourier Sine and Cosine Integrals

The function f has a Fourier representation and is *even*:
Fourier cosine integral representation of $f(x)$ on $[0,\infty)$:

$$f(x) = \int_0^\infty A(\omega)\cos\omega x d\omega$$

where

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t dt$$

is *odd*: Fourier sine integral representation of $f(x)$ on $[0,\infty)$:

$$f(x) = \int_0^\infty B(\omega)\sin\omega x d\omega \quad (1)$$

where

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t dt \quad (2)$$

Fourier Sine and Cosine Integrals (example)

Derive the Fourier cosine and sine integral of $f(x) = e^{-kx}$, where $x > 0$ and $k > 0$.

Fourier Cosine and Sine Transforms

An **integral transform** is a transformation in the form of an integral that produces from given functions new functions *depending on a different variable*. (*e.g, The Laplace transform*)

Fourier Cosine Transforms

It concerns even functions $f(x)$:

$$f(x) = \int_0^\infty A(\omega)\cos\omega x d\omega, \quad \text{where} \quad A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos\omega v dv$$

Writing $v = x$ in the formula for $A(\omega)$:

$$\boxed{\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x dx}$$

This is a transformation (**Fourier cosine transform**), which gives from $f(x)$ a new function $\hat{f}_c(\omega)$.

Fourier Cosine and Sine Transforms

$$f(x) = \int_0^\infty A(\omega) \cos \omega x d\omega, \quad \text{where} \quad A(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \cos \omega v dv$$

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos * omega x d\omega}$$

This formula gives us back $f(x)$ from $\hat{f}_c(\omega)$, and we therefore call $f(x)$ the **inverse Fourier cosine transform** of $\hat{f}_c(\omega)$.

The process of obtaining the transform $\hat{f}_c$ from a given f is also called the Fourier cosine transform or the Fourier cosine transform method.

Fourier Cosine and Sine Transforms

A similar approach can be followed for **the Fourier sine transform** and the **inverse Fourier sine transform**:

$$\boxed{\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x dx}$$

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(\omega) \sin \omega x d\omega}$$

Fourier Transform

- It is an integral transform which can be thought as similar to the Laplace transform.
- It is obtained from the Fourier integral in complex form.

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- It is an integral transform which can be thought as similar to the Laplace transform.
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The Fourier integral:

$$f(x) = \int_0^\infty (A(\omega)\cos\omega x + B(\omega)\sin\omega x)\, d\omega$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v dv$$

Fourier Transform

The resulting expression:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)e^{-i\omega v}dv\right)e^{i\omega x}d\omega$$

or

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-i\omega v}dv\right)e^{i\omega x}d\omega$$

The Fourier Transform:

$$\boxed{\mathscr{F}\{f(x)\} = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx}$$

The Inverse Fourier Transform:

$$\boxed{\mathscr{F}^{-1}\{F(\omega)\} = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega}$$

Important properties

Linearity:

$$\mathscr{F}\{af(t)+bg(t)\} = a\mathscr{F}\{f(t)\} + b\mathscr{F}\{g(t)\}$$

Shifting property:

$$\mathscr{F}\{f(t-t_0)\} = F(\omega)e^{-i\omega t_0}$$

Frequency shifting property:

$$\mathscr{F}\{e^{i\omega_0 t}f(t)\} = \int_{-\infty}^{\infty} e^{-i\omega_0 t} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{-i(\omega-\omega_0)t}dt$$

$$\mathscr{F}\{e^{i\omega_0 t}f(t)\} = F(\omega-\omega_0)$$

Fourier Transform

The Fourier Transform of derivatives:

$$\mathscr{F}\{f'(t)\} = i\omega\mathscr{F}\{f(t)\}$$

$$\mathscr{F}\{f^n(t)\} = (i\omega)^n\mathscr{F}\{f(t)\}$$

Derivative wrt ω:

$$\frac{dF(\omega)}{d\omega} = -i\mathscr{F}\{tf(t)\}$$

$$\frac{d^nF(\omega)}{d\omega^n} = (-i)^n\mathscr{F}\{t^nf(t)\}$$

Fourier Transform

The Fourier Transform of Dirac-delta function:

$$\mathscr{F}\{\delta(t-a)\} = e^{-i\omega a}$$

$$\mathscr{F}\{\delta(t)\} = 1$$

Convolution

$$(f*g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$$

$$\mathscr{F}\{(f*g)\} = \mathscr{F}\{f\}\mathscr{F}\{g\} = F(\omega)G(\omega)$$

May the Fourier be with you