Partial Differential Equations: Part I

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MMU604(MMU703) Analytical Methods in Engineering (II)

April 28, 2025

Definition

A PDE is an equation that contains one or more partial derivatives of an unknown function that depends on **at least two variables**.
Examples:

- vibrating string, membrane
- heat equation for temperature in a bar or wire
- Laplace equation for electrostatic potentials

Examples

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \rightarrow \text{1D Wave Eq.}$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \rightarrow \text{2D Heat Eq.}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \rightarrow \text{2D Laplace Eq.}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \quad \rightarrow \text{2D Poisson Eq.}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} \right) \quad \rightarrow \text{2D Wave Eq.}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \rightarrow \text{3D Laplace Eq.}$$

Wave Equation (Vibrating string)

Assumptions:

- Homogeneous string
- Perfectly elastic
- Gravitational force effect is negligible due to large axial tension
- Small transverse motion in a vertical plane (small angle approximation)

The wave equation:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

where $c^2 = \frac{T}{\rho}$.

Conditions

Boundary conditions:

$$\boxed{u(0,t) = 0, \quad u(L,t) = 0} \qquad \text{for all } t \geq 0$$

Initial conditions:

$$\boxed{u(x,0) = f(x), \quad u_t(x,0) = g(x)} \quad 0 \leq x \leq L$$

These conditions have to be satisfied!

Solution of the wave equation

Solution becomes:

$$u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^{\star} \sin \lambda_n t) \sin \frac{n\pi}{L} x$$

where $\lambda_n = cn\pi/L$ is eigenvalues (for integer n).
Considering the ICs,

$$\boxed{u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x) \quad 0 \leq x \leq L}$$

This is a Fourier sine series! Therefore:

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, ...$$

A similar approach can be developed for $B_n^{\star}$.

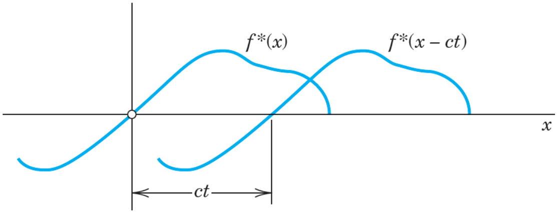
Traveling waves

The solution

$$u(x,t)=\sum_{n=1}^{\infty}B_n\cos\lambda_n t\sin\frac{n\pi x}{L},\quad \lambda_n=\frac{cn\pi}{L}$$

can be re-written in the form of

$$u(x,t)=\frac{1}{2}\left[f^{\star}(x-ct)+f^{\star}(x+ct)\right]$$



Heat Equation (Heat flow from a body in space)

Assumptions:

- Specific hear σ and density ρ of the material are constant.
- No heat is produced or disappears in the body.
- Thermal conductivity K is constant. (Homogeneous material, no extreme temperature)
- Heat flows in the direction of decreasing temperature, and the rate of flow is proportional to the gradient of the temperature.

The heat equation:

$$\boxed{\frac{\partial u}{\partial t} = c^2 \nabla^2 u}$$

where $c^2 = \frac{K}{\rho\sigma}$.

Conditions (1D heat equation, laterally insulated-long bar)

Boundary conditions:

$$\boxed{u(0,t) = 0, \quad u(L,t) = 0} \quad \text{for all } t \geq 0$$

Initial condition:

$$\boxed{u(x,0) = f(x)}$$

where $f(x)$ is given.

These conditions have to be satisfied!

Solution of the heat equation (1D)

Solution becomes:

$$u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

where $\lambda_n = cn\pi/L$ is eigenvalues (for integer n).
Considering the IC,

$$\boxed{u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x) \quad 0 \le x \le L}$$

This is a Fourier sine series! Therefore:

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \quad n = 1, 2, ...$$

Note: This is obtained for a specific BCs!

Steady 2D heat problem (Laplace's Eq.)

For steady case ($\frac{\partial u}{\partial t} = 0$), the heat equation reduces to **Laplace's equation**:

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

This is now a Boundary Value Problem that does not involve any initial conditions!