Week II: First-order Ordinary Differential Equations

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First-order ODEs

- The simplest ODEs
- Only the first derivative of the unknown function and no higher derivatives
- Can also contain unknown function y, any given function of x – $(f(x))$, and constants
- Solution: a function or a class of function

$$y = h(x) \tag{1}$$

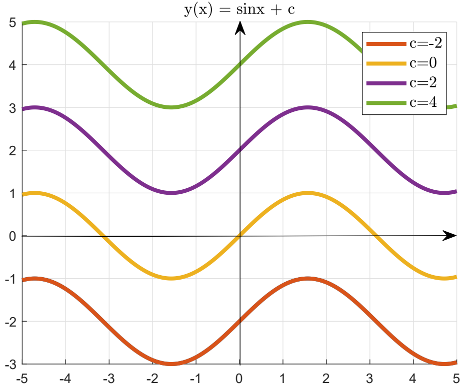
Example

ODE: $y' = cosx, \quad y(x) = ?$

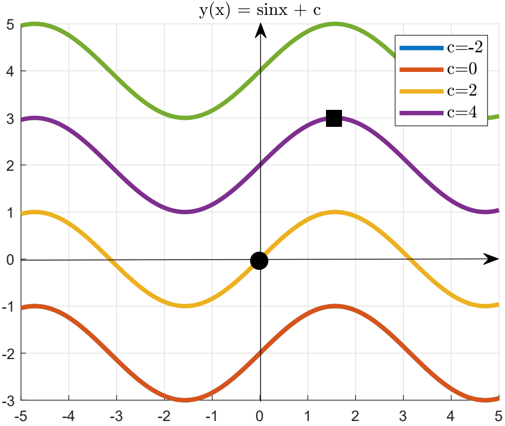
Example

ODE: $y' = cosx, \quad y(x) = ?$

Solution: $y = sinx + c \rightarrow$ General solution, c : arbitrary constant



Initial Values



Circular point $\rightarrow (0,0)$

Rectangular point $\rightarrow (\frac{\pi}{2},3)$

Example

ODE: $y' = 0.2y$

Example

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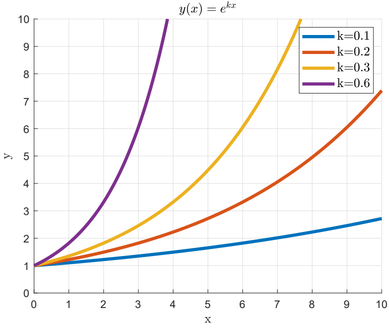
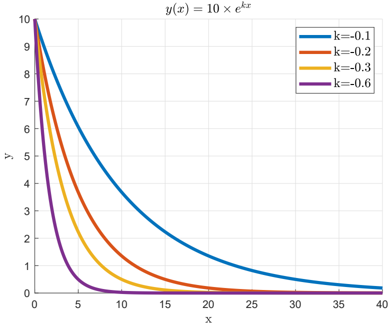
Solution: $y = ce^{0.2x}$

$0.2 \to k \qquad y = ce^{kx}$

if $k > 0$ exponential growth and if $k < 0$ exponential decay





Analytical methods for first-order ODEs

Analytical techniques can only be applied into certain types of DEs.

- Separable equations and the method of separable variables
- Exact equations
- Integrating factors
- Solution of linear ODEs

Separable equations

Many practically useful ODEs can be reduced to this form by purely algebraic manipulations

$$g(y)y' = f(x) \tag{2}$$

This is the standard form a separable equation!

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Solution can be written:

$$\int g(y)dy = \int f(x)dx + c \tag{3}$$

Examples!

Exact equations

1. Check if it is a separable equation.
2. If not, check if it is an exact equation.

The pattern of an exact equation:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{4}$$

!!!Having this pattern doesn't mean that it is an exact equation!!!

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!!!Having this pattern doesn't mean that it is an exact equation!!!

What is the benefit of having an exact equation?

If it is an exact equation, we can say that

$$\frac{d\Psi(x,y)}{dx}=0 \tag{5}$$

and the solution will be

$$\Psi(x,y)=c\rightarrow constant \tag{6}$$

Exact equations

$$\Psi(x, y) = c \rightarrow constant \tag{7}$$

What is $\Psi(x, y)$?

- It is a function of x and y.
- $\frac{\partial \Psi}{\partial x} = \Psi_x = M(x, y)$
- $\frac{\partial \Psi}{\partial y} = \Psi_y = N(x, y)$

Exact equations

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Confused?

Exact equations

What is the derivative of $\Psi(x, y(x))$ w.r.t. x? $\frac{d}{dx}\Psi(x, y(x)) = ?$
Note: y is a function of x!

Exact equations

What is the derivative of $\Psi(x, y(x))$ w.r.t. x? $\frac{d}{dx}\Psi(x, y(x)) = ?$
Applying the chain rule in partial differentiation :

$$\frac{d}{dx}\Psi(x, y) = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx} \tag{8}$$

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If $\frac{\partial \Psi}{\partial x}$ in Eq. 8 $\rightarrow M(x,y)$ in Eq. 4 and
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we know that

$$\frac{d\Psi}{dx} = 0 \tag{9}$$

and the solution

$$\boxed{\Psi(x,y) = c} \tag{10}$$

A nice property

If both Ψ_x and Ψ_y are continuous, then

$$\Psi_{xy} = \Psi_{yx} \tag{11}$$

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This property can be used to check whether it is an exact equation.

$$M_y = N_x \rightarrow \text{Exact Equation} \tag{12}$$

Note that $\Psi_x = M$ and $\Psi_y = N$.

Examples!

Integrating factors

What if it is not an exact equation?

$(3xy + y^2) + (x^2 + xy)y' = 0$

Integrating factors

What if it is not an exact equation?
What if there is a factor that makes it exact equation?
$\mu(x), \mu(y)$ or $\mu(x, y)$

Integrating factors

$P(x,y)+Q(x,y)\frac{dy}{dx}=0 \quad \rightarrow Pdx+Qdy=0$ (NOT Exact Eq.)
Assuming that the integrating factor is function of only $x \rightarrow (\mu(x))$

$$M_y=\mu P_y \quad \text{and} \quad N_x=\mu' Q+\mu Q_x \tag{13}$$

$$\mu P_y=\mu' Q+\mu Q_x \tag{14}$$

Dividing both sides by μQ

$$\frac{P_y}{Q}=\frac{d\mu}{dx}\frac{1}{\mu}+\frac{Q_x}{Q} \tag{15}$$

$$\frac{1}{\mu}\frac{d\mu}{dx}=\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=R(x) \tag{16}$$

The integration factor becomes

$$\mu(x)=e^{\int R(x)dx} \tag{17}$$

Integrating factors

Assuming that the integrating factor is function of only $y \rightarrow (\mu(y))$

$$M_y = \mu' P_y + \mu P_y \quad \text{and} \quad N_x = \mu Q_x \tag{18}$$

$$\mu Q_x = \mu' P + \mu P_y \tag{19}$$

Dividing both sides by μP

$$\frac{Q_x}{P} = \frac{d\mu}{dy}\frac{1}{\mu} + \frac{P_y}{P} \tag{20}$$

$$\frac{1}{\mu}\frac{d\mu}{dy} = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = R(y) \tag{21}$$

The integration factor becomes

$$\mu(x) = e^{\int R(y)dy} \tag{22}$$

Examples!

Linear ODEs

Standard form for a linear ODE:

$$\boxed{y' + p(x)y = r(x)} \tag{23}$$

$p(x)$ and $r(x)$ may be any given function of x.
$r(x)$: force (input)
$y(x)$: output (e.g., displacement)
If $r(x) = 0$, homogeneous:

$$y' + p(x)y = 0 \tag{24}$$

The solution is

$$\boxed{y(x) = ce^{-\int p(x)dx}} \tag{25}$$

Nonhomogenous linear ODEs

$$y' + p(x)y = r(x) \tag{26}$$

If $r(x) \neq 0$, it is a nonhomogenous linear ODE.
Taking the integrating factor $\rightarrow \mu(x)$, Eq.29 can be written as:

$$\mu y' + \mu Py = \mu R \tag{27}$$

For the left hand side

$$(\mu y)' = \mu' y + \mu y' \tag{28}$$

if $\mu Py = \mu' y \rightarrow \mu' = \mu P$ and therefore, $\mu = e^{\int Pdx} = e^h$ and $h' = P$.
Rewriting Eq.30:

$$e^h y' + e^h h' y = e^h y' + (e^h)' y = (e^h y)' = re^h \tag{29}$$

Nonhomogenous linear ODEs

$$ye^h = \int e^h r dx + c \tag{30}$$

$$y = e^{-h}\left(\int e^h r dx + c\right) \tag{31}$$

$$\boxed{y = e^{-h}\int e^h r dx + ce^{-h}} \tag{32}$$

Total output = Response to the input + Response to the initial data

Examples!

Population dynamics

Malthus

Exponential growth:

$$\frac{dP}{dt} = rP$$

Population dynamics

Malthus

Verhulst

Exponential growth:

$$\frac{dP}{dt} = rP$$

$$\rightarrow$$

Logistic equation:

$$\frac{dP}{dt} = r(1 - \frac{P}{k})P$$

Bernoulli Equation (For PhD Students)

$$y' + p(x)y = g(x)y^a$$

It's linear for $a = 0$ and $a = 1$, otherwise nonlinear.

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With this, the solution of the logistic equation is possible:

$$\text{Logistic equation:} y' = Ay - By^2$$

Logistic Equation (For PhD Students)

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The solution:

$$y = \frac{1}{u} = \frac{1}{ce^{-At} + B/A}$$

End of this week!
Next week: Second-order ODEs