Week IV: Higher Order Linear ODEs & Systems of ODEs

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MMU604(MMU703) Analytical Methods in Engineering (II)

March 10, 2025

Higher order linear ODEs

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$r \neq 0 \rightarrow$ nonhomogeneous

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$y \rightarrow$ basis

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- Linear independence (Wronskian)

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0 (\text{ linearly independent}) \tag{5}$$

Initial Value Problem

- 1^{st} order $\rightarrow$ one initial condition
- 2^{nd} order $\rightarrow$ two initial conditions
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- Example (Third order Euler-Cauchy Equation), three roots!

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- Simple complex roots
- Multiple roots
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- Simple complex roots
- Multiple roots
- Multiple complex roots
- Example for each

Nonhomogeneous Linear ODEs

$$y^n + p_{n-1}(x)y^{n-1} + \ldots + p_1(x)y' + p_0(x)y = r(x) \qquad (8)$$

where $r \neq 0$.

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General solution:

$$y(x) = y_h(x) + y_p(x) \tag{9}$$

y_h : solution of the homogeneous part
y_p : any solution of the nonhomogeneous equation without arbitrary constants

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Input $r(x)$	**Solution** $y_p(x)$
ke^{ax}	Ce^{ax}
kx^n	$K_nx^n + ... + K_1x + K_0$
$kcos\omega x$	$Kcos\omega x + Msin\omega x$
$ksin\omega x$	$Kcos\omega x + Msin\omega x$

Method of variation of parameters

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For $2^n d$ order:

$$y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx \qquad (10)$$

For $\mathrm{n}^t h$ order:

$$y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)} r(x) dx$$

where W_j is obtained from W by replacing the jth column of W by the column $[0, 0, ..., 1]^T$.

Systems of ODEs

- Systems governed by a series of ODEs

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- Higher order ODEs can be reduced to a series of 1^{st} order ODEs and be solved!

Systems of ODEs

Example:

$$y_1' = -0.02y_1 + 0.02y_2$$
$$y_2' = 0.02y_1 - 9.92y_2$$

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$$\mathbf{y}' = \mathbf{A}\mathbf{y} \tag{11}$$

where

$$A = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix} \tag{12}$$

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Systems of ODEs

From the superposition principal, the general solution:

$$y = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t} \tag{15}$$

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The particular solution can be obtained using the initial conditions $(y_1(0) = 0, y_2(0) = 150)$:

$$y_1 = 75 - 75e^{-0.04t}$$
$$y_2 = 75 + 75e^{-0.04t}$$

General system of ODEs

More general system can be written

$$
\begin{aligned}
y_1' &= f_1(t, y_1, ..., y_n) \\
y_2' &= f_2(t, y_1, ..., y_n) \\
&... \\
y_n' &= f_n(t, y_1, ..., y_n)
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$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}) \tag{16}$$

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For instance, if n=1:

$$y_1' = f_1(t, y_1) \tag{17}$$

and the solution will be

$$y_1 = h_1(t) \tag{18}$$

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The solution for the system of ODE can be expressed:

$$\mathbf{y} = \mathbf{h}(t) \tag{17}$$

Linear system

Linear system (consisting of linear ODEs):

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\begin{aligned}
y_1' &= a_{11}(t)y_1 + \ldots + a_{1n}(t)y_n + g_1(t) \\
&\ldots \\
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$$\boxed{\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}} \tag{18}$$

If $g = 0 \rightarrow$ homogeneous
If $g \neq 0 \rightarrow$ nonhomogeneous

Linear system

General solution

$$\mathbf{y} = c_1\mathbf{y}^{(\mathbf{1})} + \ldots + c_n\mathbf{y}^{(\mathbf{n})} \tag{19}$$

where **y** is basis.

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where $\mathbf{y}$ is basis.

$$\mathbf{Y} = \left[\mathbf{y}^{(\mathbf{1})}...\mathbf{y}^{(\mathbf{n})}\right] \tag{20}$$

The determinant of $\mathbf{Y} \rightarrow$ Wronskian:

$$W(y^{(1)}, ..., y^{(n)}) = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & ... & y_1^{(n)} \\ ... & & & \\ y_n^{(1)} & y_n^{(2)} & ... & y_n^{(n)} \end{vmatrix} \tag{21}$$

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If $W \neq 0$

$$\mathbf{y} = \mathbf{YC} \tag{22}$$

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$$\mathbf{y}' = \mathbf{A}\mathbf{y} \tag{23}$$

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Eigenvalue problem

$$\boxed{\mathbf{A}\mathbf{x} = \lambda \mathbf{x}} \tag{26}$$

The basis:

$$\mathbf{y}^{(\mathbf{1})} = \mathbf{x}^{(\mathbf{1})}e^{\lambda_1 t}, \quad \dots, \quad \mathbf{y}^{(\mathbf{n})} = \mathbf{x}^{(\mathbf{n})}e^{\lambda_n t} \tag{27}$$

Nonhomogeneous linear system of ODEs

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \tag{28}$$

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The solution

$$\mathbf{y} = \mathbf{y_h} + \mathbf{y_p} \tag{29}$$

Method of undetermined coefficients

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \tag{30}$$

A particular solution $y^{(p)}$ is assumed in a form similar to g.

•Example

Method of variation of parameters (For PhD Students)

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{g}(t) \tag{31}$$

To apply the method, the particular solution is assumed as:

$$\mathbf{y}^{(\mathbf{p})} = \mathbf{Y}(t)\mathbf{u}(t) \tag{32}$$

Take the derivative and substitute into above solution to obtain $\mathbf{u}(t)$.

End of this week.