# Week IV: Higher Order Linear ODEs & Systems of ODEs

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$$y^{(n)} = rac{d^n y}{dx^n} 
ightarrow n^{th}$$
order, which is the highest power

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(1)

$$y^{(n)} = \frac{d^n y}{dx^n} \to n^{th}$$
 order, which is the highest power (1)

• High order: n > 2 $F(x, y, y', ..., y^{(n)}) = 0$  (2)

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• High order: 
$$n > 2$$
  
 $F(x, y, y', ..., y^{(n)}) = 0$  (2)

Standard form of linear ODE:

$$y^{(n)} + p_{n-1}(x)y^{n-1} + \dots + p_1(x)y' + p_0(x)y = r(x)$$
(3)

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 $r = 0 \rightarrow$  homogeneous  $r \neq 0 \rightarrow$  nonhomogeneous

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#### Homogeneous Linear ODEs

• Superposition principle can be extended to higher order homogeneous ODEs

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## Homogeneous Linear ODEs

- Superposition principle can be extended to higher order homogeneous ODEs
- A general solution

$$y(x) = c_1 y_1(x) + ... + c_n y_n(x)$$
 (4)

 $c \rightarrow$  arbitrary constants  $y \rightarrow$  basis

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## Homogeneous Linear ODEs

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 (4)

- $c \rightarrow$  arbitrary constants  $v \rightarrow$  basis
- Linear independence (Wronskian)

$$W = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y'_1 & y'_2 & \dots & y'_n \\ \dots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \neq 0 (\text{ linearly independent}) \quad (5)$$

3/20

## Initial Value Problem

- $1^{st}$  order ightarrow one initial condition
- $2^{nd}$  order  $\rightarrow$  two initial conditions
- $n^{th}$  order  $\rightarrow$  n initial conditions

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#### Initial Value Problem

- $1^{st}$  order  $\rightarrow$  one initial condition
- $2^{nd}$  order  $\rightarrow$  two initial conditions
- $n^{th}$  order  $\rightarrow$  n initial conditions
- Example (Third order Euler-Cauchy Equation), three roots!

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$$y^{n} + a_{n-1}y^{n-1} + \dots + a_{1}y' + a_{0}y = 0$$
(6)

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$$y^{n} + a_{n-1}y^{n-1} + \dots + a_{1}y' + a_{0}y = 0$$
(6)

Characteristic equation:

$$\left|\lambda^{n} + a_{n-1}\lambda^{n-1} + \ldots + a_{1}\lambda + a_{0}\lambda = 0\right|$$
(7)

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- Distinct real roots
- Simple complex roots
- Multiple roots
- Multiple complex roots

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- Distinct real roots
- Simple complex roots
- Multiple roots
- Multiple complex roots
- Example for each

#### Nonhomogeneous Linear ODEs

$$y^{n} + p_{n-1}(x)y^{n-1} + \dots + p_{1}(x)y' + p_{0}(x)y = r(x)$$
(8)

where  $r \neq 0$ .

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## Nonhomogeneous Linear ODEs

$$y^{n} + p_{n-1}(x)y^{n-1} + \dots + p_{1}(x)y' + p_{0}(x)y = r(x)$$
(8)

where  $r \neq 0$ . General solution:

$$y(x) = y_h(x) + y_p(x)$$
(9)

 $y_h$ : solution of the homogeneous part  $y_p$ : any solution of the nonhomogeneous equation without arbitrary constants

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## Nonhomogeneous Linear ODEs

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 $y_{\it p}$  : any solution of the nonhomogeneous equation without arbitrary constants

Input $r(x)$	Solution $y_p(x)$			
ke <sup>ax</sup>	Ce <sup>ax</sup>			
<i>kx</i> <sup>n</sup>	$K_n x^n + \ldots + K_1 x$	$+K_0$		
kcosωx	$Kcos\omega x + Msin\omega$	Х		
ksinωx	$Kcos\omega x + Msin\omega$	X		
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#### Method of variation of parameters

• The method of undetermined coefficients is suitable for linear ODEs with **constant coefficients**.

7/20

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#### Method of variation of parameters

- The method of undetermined coefficients is suitable for linear ODEs with **constant coefficients**.
- The method of variation of parameters can be applied.

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- The method of undetermined coefficients is suitable for linear ODEs with **constant coefficients**.
- The method of variation of parameters can be applied.

For  $2^n d$  order:

$$y_{p}(x) = -y_{1} \int \frac{y_{2}r}{W} dx + y_{2} \int \frac{y_{1}r}{W} dx$$
(10)

For n<sup>t</sup> h order:

$$y_p(x) = \sum_{k=1}^n y_k(x) \int \frac{W_k(x)}{W(x)} r(x) dx$$

where  $W_j$  is obtained from W by replacing the *jth* column of W by the column  $[0, 0, ..., 1]^T$ .

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• Systems governed by a series of ODEs

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- Systems governed by a series of ODEs
- Higher order ODEs can be reduced to a series of 1<sup>st</sup> order ODEs and be solved!

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Example:

$$y_1' = -0.02y_1 + 0.02y_2$$
$$y_2' = 0.02y_1 - 9.92y_2$$

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#### Example:

$$y_1' = -0.02y_1 + 0.02y_2$$
  
$$y_2' = 0.02y_1 - 9.92y_2$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \tag{11}$$

where

$$A = \begin{bmatrix} -0.02 & 0.02\\ 0.02 & -0.02 \end{bmatrix}$$
(12)

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A general solution:

$$\mathbf{y} = \mathbf{x} e^{\lambda t} \tag{13}$$

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$$\mathbf{y}' = \lambda \mathbf{x} e^{\lambda t} \tag{14}$$

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9/20

From the superposition principal, the general solution:

$$y = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}$$
(15)

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10 / 20

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(15)

The particular solution can be obtained using the initial conditions  $(y_1(0) = 0, y_2(0) = 150)$ :

$$y_1 = 75 - 75e^{-0.04t}$$
  
$$y_2 = 75 + 75e^{-0.04t}$$

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More general system can be written

$$y'_1 = f_1(t, y_1, ..., y_n)$$
  
 $y'_2 = f_2(t, y_1, ..., y_n)$   
...

$$y'_n = f_n(t, y_1, ..., y_n)$$

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More general system can be written

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$$y'_{2} = f_{2}(t, y_{1}, ..., y_{n})$$
  
...  

$$y'_{n} = f_{n}(t, y_{1}, ..., y_{n})$$

Therefore,

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}) \tag{16}$$

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11 / 20

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Therefore,

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For instance, if n=1:

$$y_1' = f_1(t, y_1)$$
 (17)

and the solution will be

$$y_1 = h_1(t) \tag{18}$$

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More general system can be written

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$$y'_{2} = f_{2}(t, y_{1}, ..., y_{n})$$
  
...  

$$y'_{n} = f_{n}(t, y_{1}, ..., y_{n})$$

Therefore,

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}) \tag{16}$$

The solution for the system of ODE can be expressed:

$$\mathbf{y} = \mathbf{h}(t) \tag{17}$$

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### Linear system

Linear system (consisting of linear ODEs):

$$y'_{1} = a_{11}(t)y_{1} + \dots + a_{1n}(t)y_{n} + g_{1}(t)$$
  
...  
$$y'_{n} = a_{n1}(t)y_{1} + \dots + a_{nn}(t)y_{n} + g_{n}(t)$$

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#### Linear system

Linear system (consisting of linear ODEs):

$$y'_{1} = a_{11}(t)y_{1} + \dots + a_{1n}(t)y_{n} + g_{1}(t)$$
  
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$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \tag{18}$$

If  $g = 0 \rightarrow$  homogeneous If  $g \neq 0 \rightarrow$  nonhomogeneous

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## Linear system

General solution

$$\mathbf{y} = c_1 \mathbf{y}^{(1)} + \dots + c_n \mathbf{y}^{(n)}$$
(19)

where **y** is basis.

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#### Linear system

General solution

$$\mathbf{y} = c_1 \mathbf{y}^{(1)} + \ldots + c_n \mathbf{y}^{(n)} \tag{19}$$

where **y** is basis.

$$\mathbf{Y} = \begin{bmatrix} \mathbf{y}^{(1)} \dots \mathbf{y}^{(n)} \end{bmatrix}$$
(20)

The determinant of  $\textbf{Y} \rightarrow \text{Wronskian}:$ 

$$W(y^{(1)},...,y^{(n)}) = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & \dots & y_1^{(n)} \\ \dots & & & \\ y_n^{(1)} & y_n^{(2)} & \dots & y_n^{(n)} \end{vmatrix}$$
(21)

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#### Linear system

General solution

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(21)

If  $W \neq 0$ 

$$\mathbf{y} = \mathbf{Y}\mathbf{C} \tag{22}$$

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## Constant coefficient systems

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \tag{23}$$

where A does not depends on t.

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### Constant coefficient systems

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \tag{23}$$

where A does not depends on t.  $y' = ky \rightarrow y = Ce^{kt}$ 

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### Constant coefficient systems

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \tag{23}$$

where A does not depends on t.  $y' = ky \rightarrow y = Ce^{kt}$ 

$$\mathbf{y} = \mathbf{x} e^{\lambda t} \tag{24}$$

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### Constant coefficient systems

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$$\mathbf{y} = \mathbf{x} e^{\lambda t} \tag{24}$$

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$$\mathbf{y}' = \lambda \mathbf{x} e^{\lambda t} = \mathbf{A} \mathbf{x} e^{\lambda t} \tag{25}$$

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## Constant coefficient systems

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where A does not depends on t.  $y' = ky \rightarrow y = Ce^{kt}$ 

$$\mathbf{y} = \mathbf{x} e^{\lambda t} \tag{24}$$

$$\mathbf{y}' = \lambda \mathbf{x} e^{\lambda t} = \mathbf{A} \mathbf{x} e^{\lambda t}$$
(25)

Eigenvalue problem

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \tag{26}$$

The basis:

$$\mathbf{y}^{(1)} = \mathbf{x}^{(1)} e^{\lambda_1 t}, \quad ..., \quad \mathbf{y}^{(n)} = \mathbf{x}^{(n)} e^{\lambda_n t}$$
 (27)

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14 / 20

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#### Nonhomogeneous linear system of ODEs

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \tag{28}$$

where  $g \neq 0$ .

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# Nonhomogeneous linear system of ODEs

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \tag{28}$$

where  $g \neq 0$ . The solution

$$\mathbf{y} = \mathbf{y}_{\mathbf{h}} + \mathbf{y}_{\mathbf{p}} \tag{29}$$

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15 / 20

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## Method of undetermined coefficients

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \tag{30}$$

A particular solution  $y^{(p)}$  is assumed in a form similar to g.

•Example

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16 / 20

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#### Method of variation of parameters (For PhD Students)

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{g}(t) \tag{31}$$

To apply the method, the particular solution is assumed as:

$$\mathbf{y}^{(\mathbf{p})} = \mathbf{Y}(t)\mathbf{u}(t) \tag{32}$$

Take the derivative and substitute into above solution to obtain  $\mathbf{u}(t)$ .

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End of this week.

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