Week V: Series Solution of ODEs

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Legendre, Frobenius and Bessel

Adrien-Marie Legendre

Ferdinand Georg Frobenius

Friedrich Bessel

Power series

- Legendre's equations
- Bessel's equations

Power series:

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \dots \tag{1}$$

where x_0 is the centre of the series.
For $x_0 = 0$,

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \dots \tag{2}$$

Power series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \ldots \tag{3}$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \tag{4}$$

$$cosx = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \tag{5}$$

$$sinx = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \tag{6}$$

Example:

$$y' - y = 0 \tag{7}$$

We know that solution in the form:

$$y = ce^x \tag{8}$$

Now, have the same solution by using the power series.

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \dots \tag{9}$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots \tag{10}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^m = 2a_2 + 6a_3^2 + 12a_4^3 + \dots \tag{11}$$

Method in general

Standard form:

$$y'' + P(x)y' + Q(x)y = 0 \tag{12}$$

- Express $P(x)$ and $Q(x)$ in power series if they are not polynomial
- Collect same powers of x and equate the sum of the coefficients of each power of x to zero.
- Determine unknown coefficients $(a_0, a_1, ...)$

Convergence

n^{th} partial sum:

$$s_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + a_n(x - x_0)^n \tag{13}$$

the rest is the remainder:

$$R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \ldots \tag{14}$$

The sequence converges if $\lim_{n\to\infty} s_n(x_1) = s(x_1)$.
In the case of convergence:

$$|R_n(x)| = |s(x_1) - s_n(x)| < \varepsilon \tag{15}$$

More about convergence

R: Radius of convergence
The series converges for all x

$$|x - x_0| < R \to \text{converges}$$
$$|x - x_0| > R \to \text{diverges}$$

If $R = \infty$, convergence for all x (best possible case), If $R = 0$, convergence only at the centre (useless).

Legendre's Equation

Legendre's differential equation:

$$\left(1-x^2\right) y'' - 2xy' + n(n+1)y = 0 \tag{16}$$

Written in the standard form:

$$y'' - \frac{2x}{1-x^2} y' + \frac{n(n+1)}{1-x^2} y = 0 \tag{17}$$

$P(x)$ and $Q(x)$ are analytic at $x = 0$.
Assume the solution as $y = \sum_{m=0}^{\infty} a_m x^m$.

$$a_2 = \frac{n(n+1)}{2} a_0, \quad a_3 = \frac{2+n(n+1)}{6} a_1 \tag{18}$$

Legendre's Equation

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \tag{19}$$

Recurrence relation.

$$a_4 = -\frac{(n-2)(n+3)}{4 \times 3} a_2, \quad a_5 = -\frac{(n-3)(n+4)}{5 \times 4} a_3 \tag{20}$$

Solution:

$$y(x) = \sum_{m=0}^{\infty} a_m x^m \to y(x) = a_0 y_1(x) + a_1 y_2(x) \tag{21}$$

$$y_1(x) = 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} x^4 - \dots \tag{22}$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \dots \tag{23}$$

Legendre's Polynomial

For non-negative integer n values, either $y_1(x)$ or $y_2(x)$ becomes finite (the series terminates).
The coefficient is obtained:

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-m)!}$$

Using the above expression, the resulting solution of Legendre's DE is called the Legendre Polynomial of degree n and is denoted by $P_n(x)$

$$\begin{aligned} P_n(x) &= \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-m)!} x^{n-2m} \\ &= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + \ldots \end{aligned}$$

Frobenius Method

Standard form:

$$y'' + P(x)y' + Q(x)y = 0 \tag{24}$$

$P(x)$ and $Q(x)$ are analytic at $x = 0$.
Even if the coefficients are not analytic, they can still be solved by the series.

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0 \tag{25}$$

$b(x)$ and $c(x)$ have to be analytic at $x = 0$.
Solution:

$$y = x^r \sum_{m=0}^{\infty} a_m x^m = x^r(a_0 + a_1 x + a_2 x^2 + \ldots) \tag{26}$$

r is chosen so that $a_0 \neq 0$.

Frobenius Method

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0 \tag{27}$$

$$x^2y'' + xb(x)y' + c(x)y = 0 \tag{28}$$

- Expand $b(x)$ and $c(x)$ in power series $(b_0 + b_1x + ...)$ and $(c_0 + c_1x + ...)$, if they are not polynomial.
- Collect the same powers of x and equate the sum of the coefficients of each power of x to zero.
- Solve *the indicial equation* to determine its roots.

Frobenius Method

Solution:

$$y = x^r \sum_{m=0}^{\infty} a_m x^m = x^r(a_0 + a_1 x + a_2 x^2 + \dots) \tag{29}$$

$$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{(m+r-1)} = x^{r-1}(r a_0 + (r+1) a_1 x + \dots) \tag{30}$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{(m+r-2)} = x^{r-2}(r(r-1)a_0 + (r+1) r a_1 x + \dots) \tag{31}$$

Frobenius Method

Substituting the terms into the ODE yields:

$$x^r \to [r(r-1) + b_0 r + c_0]a_0 = 0$$

which is called the *Indicial equation*.
There are 3 possible cases:

- Case 1: distinct roots not differing by an integer
- Case 2: double root
- Case 3: roots differing by an integer

Bessel's Equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{32}$$

where $\nu \geq 0$ (real number). The solution:

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \quad (a_0 \neq 0) \tag{33}$$

Then, the indicial equation can be written:

$$(r+\nu)(r-\nu) = 0 \tag{34}$$

The roots are $r_1 = \nu(\geq 0)$ and $r_2 = -\nu$.

Bessel's Equation

For $r = r_1 = \nu$, it is found that $a_1 = 0, a_3 = 0, a_5 = 0 \ldots$. Only even-numbered coefficients are non-zero and a recursion formula can be derived:

$$a_{2m} = \frac{(-1)^m a_o}{2^{2m} m!(\nu+1)(\nu+2)\ldots(\nu+m)} \quad m = 1, 2, \ldots \tag{35}$$

where a_0 is arbitrary constant.
For integer values of ν (n)

$$a_{2m} = \frac{(-1)^m a_o}{2^{2m} m!(n+1)(n+2)\ldots(n+m)} \quad m = 1, 2, \ldots \tag{36}$$

Bessel's Equation

The solution can be obtained as

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!} \quad n \geq 0 \tag{37}$$

$J_n(x)$ is called the Bessel function of the first kind of order n.

Bessel's Equation

Bessel function $J_\nu(x)$ for any $\nu \geq 0$:

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m! \Gamma(\nu+m+1)} \tag{38}$$

$J_\nu(x)$ is called the Bessel function of the first kind of order ν.
Bessel function $J_\nu(x)$ with half-integer ν:

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$
$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Bessel's Equation

For a general solution, a second linearly independent solution is required. If ν is not an integer, by replacing ν by $-\nu$ provides the second solution:

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m! \Gamma(m-\nu+1)} \tag{39}$$

Therefore, the general solution:

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x) \tag{40}$$

However, for any integer $\nu = n$, solutions become linearly dependent:

$$J_{-n}(x) = (-1)^n J_n(x) \quad n = 1, 2, \ldots \tag{41}$$

Bessel's Equation

Bessel function of the second kind $Y_{\nu}(x)$:

$$Y_{\nu}(x) = \frac{1}{\sin \nu\pi}[J_{\nu}(x)\cos \nu\pi - J_{-\nu(x)}] \tag{42}$$

or for integer value:

$$Y_n(x) = \frac{2}{\pi}J_n(x)\left(ln\frac{x}{2} + \gamma\right) + \frac{x^n}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}(h_m + h_{m+n}}{2^{2m+n}m!(m+n)!}x^{2m}$$
$$-\frac{x^{-n}}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{2^{2m-n}m!}x^{2m}$$

A general solution of Bessel's equation for all values of ν is

$$y(x) = C_1 J_{\nu}(x) + C_2 Y_{\nu}(x) \tag{43}$$