Laplace Transforms, eigenvalue problem, grad, div, and curl

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MMU604(MMU703) Analytical Methods in Engineering (II)

March 24, 2025

Laplace Transforms

- More directly without determining a general solution/homogeneous ODE
- Inputs that have discontinuities or represent short impulses or complicated periodic functions (e.g., the unit step function, the Dirac's delta)

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Steps:

1. Apply the Laplace Transform (LT) $\rightarrow$ algebraic equation
2. Solve the equation by purely algebraic manipulation
3. Transform back into time-domain

Laplace Transforms

The Laplace Transform:

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \tag{1}$$

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The Inverse Transform:

$$f(t) = \mathscr{L}^{-1}(F) \tag{2}$$

Table 6.1 **Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$**

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\dfrac{s-a}{(s-a)^2 + \omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2 + \omega^2}$

s-Shifting

$$F(s-a)=\int_0^\infty e^{-(s-a)t}f(t)dt=\int_0^\infty e^{-st}e^{at}f(t)dt$$

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If we know $f(t)$, we can find $e^{at} f(t)$.

Transforms of derivatives and integrals

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)dt = \left[e^{-st} f(t)\right]\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt \quad (3)$$

$$= 0 - 1.f(0) + s\int_0^\infty e^{-st} f(t)dt \quad (4)$$

$$= s\mathscr{L}\left\{f(t)\right\} - f(0) \quad (5)$$

Transforms of derivatives and integrals

$$\mathscr{L}\left\{f''(t)\right\} = \int_0^\infty e^{-st} f''(t)dt = \left[e^{-st} f'(t)\right]\Big|_0^\infty + s\int_0^\infty e^{-st} f'(t)dt \tag{6}$$

$$= 0 - 1.f'(0) + s\int_0^\infty e^{-st} f'(t)dt \tag{7}$$

$$= s^2 \mathscr{L}\left\{f(t)\right\} - sf(0) - f'(0) \tag{8}$$

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Derivative:

$$\mathscr{L}\left\{f^{(n)}\right\} = s^n \mathscr{L}\left\{f\right\} - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{n-1}(0) \quad (9)$$

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Integral:

$$\mathscr{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} F(s) \quad (10)$$

$$y'' + ay' + by = r(t), \qquad y(0) = K_0, y'(0) = K_1 \tag{11}$$

$$s^2 Y(s) - sy(0) - y'(0) + a[sY(s) - y(0)] + bY(s) = R(s) \tag{12}$$

For zero initial conditions:

$$\frac{Y(s)}{R(s)} = \frac{1}{s^2 + as + b} \rightarrow G(s) \quad \text{Transfer function} \tag{13}$$

The response:

$$Y(s) = \left[(s+a)\,y(0) + y'(0)\right] G(s) + R(s)G(s) \tag{14}$$

Unit step function

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The Laplace Transform:

$$\mathscr{L}\{u(t-a)\} = \int_0^\infty u(t-a)e^{-st}dt \tag{16}$$

$$= \int_0^\infty e^{-st}1dt = -\frac{e^{-st}}{s}\bigg|_{t=a}^{\infty}, \quad t = a \leq 0 \tag{17}$$

$$\boxed{\mathscr{L}\{u(t-a)\} = \frac{e^{-as}}{s}} \tag{18}$$

where $s > 0$.

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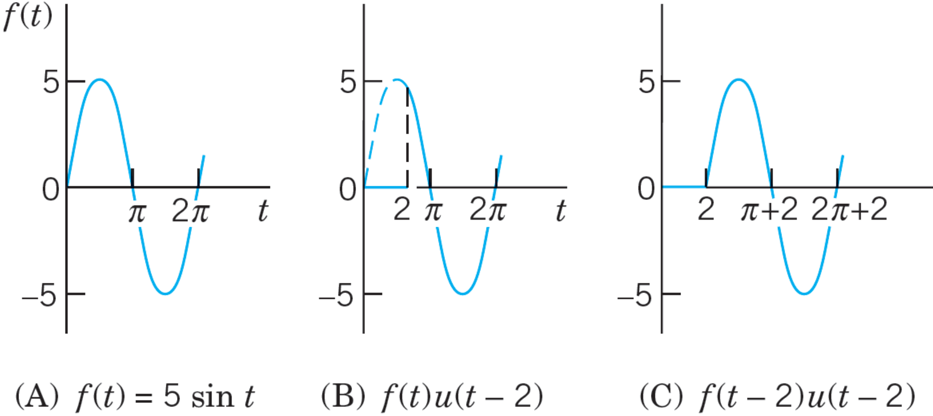
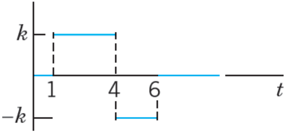
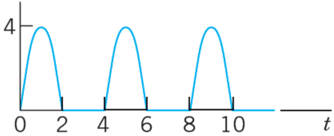
On-off switch!


f(t)
5
0
–5
π
2π
t
5
0
–5
2
π
2π
t
5
0
–5
2
π+2
2π+2
(A) f(t) = 5 sin t
(B) f(t)u(t – 2)
(C) f(t – 2)u(t – 2)

[]



(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$



(B) $4 \sin(\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

[]

t-Shifting

Replacing t by $(t-a)$ in $f(t)$.

$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if} \quad t < a \\ f(t-a) & \text{if} \quad t > a \end{cases} \tag{19}$$

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The Laplace Transform:

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau} f(\tau)d\tau \tag{20}$$

$$= \int_0^\infty e^{-s(\tau+a)} f(\tau)d\tau \tag{21}$$

$$= \int_a^\infty e^{-st} f(t-a)dt \tag{22}$$

$$= \int_0^\infty e^{-st} f(t-a)u(t-a)dt = \int_0^\infty e^{-st}\tilde{f}(t)dt \tag{23}$$

t-Shifting

The Laplace Transform:

$$\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) \tag{24}$$

Dirac's Delta Function

A function:

$$f_k(t-a) = \begin{cases} 1/k & \text{if} \quad a \leq t \leq a+k \\ 0 & otherwise \end{cases} \tag{25}$$

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$$\delta(t-a) = \lim_{k \to 0} f_k(t-a) \tag{27}$$

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$$\delta(t-a) = \begin{cases} \infty & \text{if} \quad t = a \\ 0 & otherwise \end{cases} \tag{28}$$

and the area:

$$\int_0^\infty \delta(t-a)dt = 1 \tag{29}$$

Dirac's Delta Function

$$f_k(t-a) = \frac{1}{k}\left[u(t-a) - u(t-(a+k))\right] \tag{30}$$

The Laplace Transform:

$$\mathscr{L}\{f_k(t-a)\} = \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right] = e^{-as}\frac{1-e^{-ks}}{ks} \tag{31}$$

taking the limit as $k \to 0$ (L'Hopital):

$$\frac{+se^{-ks}}{s} \to 1 \quad \text{as} \quad k \to 0 \tag{32}$$

Hence,

$$\mathscr{L}\{f_k(t-a)\} = e^{-as} \tag{33}$$

Convolution

$\mathscr{L}\{f\}\mathscr{L}\{g\}$ is the transform of the convolution of f and g.

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau \tag{34}$$

where $h(t) = \mathscr{L}^{-}1\{H\}$ and $H = FG$.

Convolution

A nonhomogeneous linear ODE:

$$y'' + ay' + by = r(t) \tag{35}$$

$$Y(s) = \left[(s+a)y(0) + y'(0)\right] G(s) + R(s)G(s) \tag{36}$$

For zero initial values:

$$Y(s) = R(s)G(s) \tag{37}$$

The convolution theorem gives the solution:

$$\boxed{y(t) = \int_0^t g(t-\tau)r(\tau)d\tau} \tag{38}$$

Eigenvalue Problem

Eigenvalue problem:

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \tag{39}$$

$\boldsymbol{A}$: a given non-zero square matrix
$\boldsymbol{x}$: an unknown vector (eigenvector)
$\boldsymbol{\lambda}$: an unknown scalar (eigenvalue)

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Find λ and x satisfying the above equation.

Eigenvalue Problem

$$\begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 33 \\ 27 \end{bmatrix} \tag{40}$$

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$$\boldsymbol{Ax} = \lambda \boldsymbol{x} \tag{42}$$

Example!

Eigenvalue Problem

$$
\begin{aligned}
a_{11}x_1 + \ldots + a_{1n}x_n &= \lambda x_1 \\
a_{21}x_1 + \ldots + a_{2n}x_n &= \lambda x_2 \\
&\ldots \\
a_{n1}x1 + \ldots + a_{nn}x_n &= \lambda x_n
\end{aligned}
$$

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&\ldots \\
a_{n1}x1 + \ldots + a_{nn}x_n &= \lambda x_n
\end{aligned}
$$

$$
\begin{aligned}
(a_{11} - \lambda)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\
a_{21}x_1 + (a_{22} - \lambda)x_2 + \ldots + a_{2n}x_n &= 0 \\
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a_{n1}x_1 + a_{n2}x_2 + \ldots + (a_{nn} - \lambda)x_n &= 0
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In matrix notation:

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = 0$$

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Characteristic equation can be written:

$$D(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I})$$

Vector calculus

gradiend, divergence, and curl