Fourier Analysis: Part I

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Fourier Series

Three ways of showing the Fourier Series:

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

$$f(x) = \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

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How do we find the coefficients?

Fourier Coefficients

The answer is the orthogonality!

$$\int_{-\pi}^{\pi} (\cos nx)(\cos mx)dx = 0 \quad (m \neq n)$$

$$\int_{-\pi}^{\pi} (\sin nx)(\sin mx)dx = 0 \quad (m \neq n)$$

$$\int_{-\pi}^{\pi} (\sin nx)(\cos mx)dx = 0 \quad (m \neq n) \quad \text{or} \quad (m = n)$$

Fourier Coefficients

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

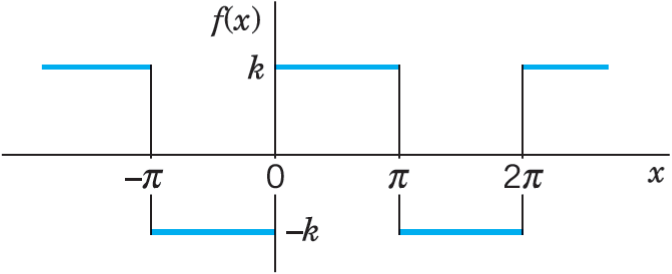
After the derivation using the orthogonality, the coefficients are found:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Example: Find the Fourier coefficients of the periodic function $f(x)$ in the figure.



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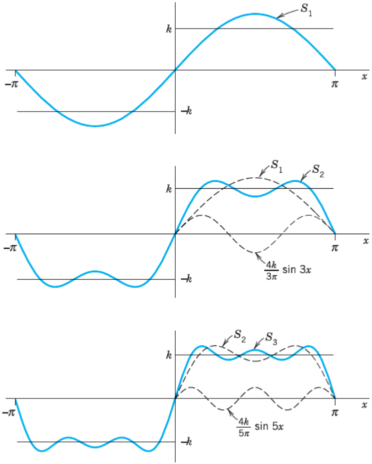

Fourier Series (any period p=2L)

A new variable is introduced:

$$x = \frac{P}{2\pi} v, \quad \text{where} \quad p = 2L$$

The series is written as:

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)}$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

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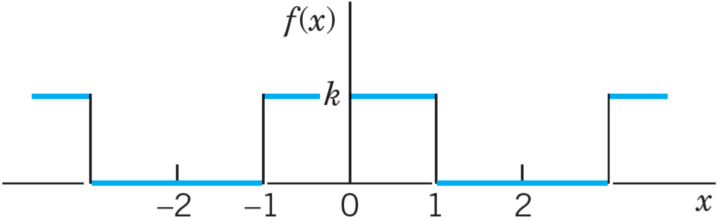
Fourier Series expansion can be only applied into periodic functions.

Example: Find the Fourier coefficients of the periodic function $f(x)$ in the figure.



Simplifications: Even and Odd Functions

If the function $f(x)$ is **even** (like cos), the coefficients of sin terms are 0.

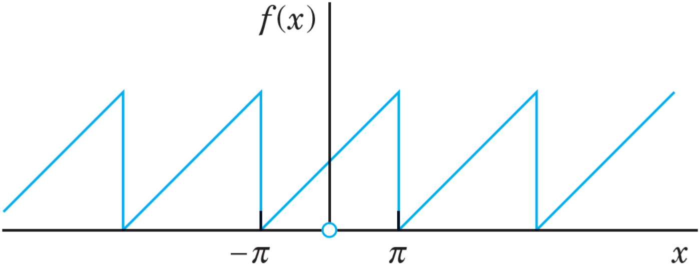
$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

If the function $f(x)$ is **odd** (like sin), the coefficients of cos terms are 0.

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

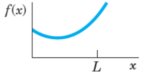
Attention: The coefficients re multiplied by the factor of 2. The integral is from 0 to L.

Example: Find the Fourier coefficients of the periodic function $f(x)$ in the figure.

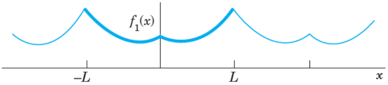


Half-Range Expansion

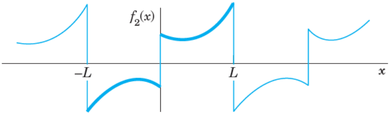
A function $f(x)$ as a function of period L can be extended and developed the extended function into a Fourier series.



(0) The given function $f(x)$

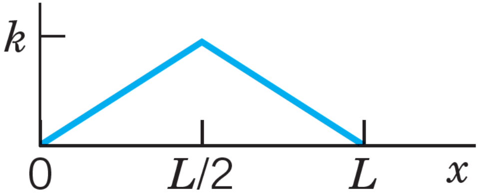


(a) $f(x)$ continued as an ***even*** periodic function of period $2L$



(b) $f(x)$ continued as an ***odd*** periodic function of period $2L$

Example: Find the two half-range expansions of the function in the figure.



Forced Oscillations (practical example

Example: An excitation in the figure below is applied into a mass-damper-spring system.