BESSEL'S EQUATION AND BESSEL FUNCTIONS:

The differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - n^2\right) y = 0 \qquad (1)$$

where n is a parameter, is called *Bessel's Equation of Order n*.

Any solution of *Bessel's Equation of Order n* is called a *Bessel Function of Order n*.

Bessel’s Equation and Bessel’s Functions occur in connection with many problems of physics and engineering, and there is an extensive literature dealing with the theory and application of this equation and its solutions.

If n=0 Equation (1) is equivalent to the equation

$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + xy = 0 \qquad (2)$$

which is called *Bessel's Equation of Order Zero*.

The general Solution of Equation (2) is given by :

$$y = A\,J_0(x) \;+ B\,Y_0(x)$$

where A and B are arbitrary constants,

and J_0 is called the *Bessel Function of the First Kind of Order Zero*.

Y_0 is called the *Bessel Function of the Second Kind of Order Zero*.

The functions J_0 and Y_0 have been studied extensively and tabulated. Many of the interesting properties of these functions are indicated by their graphs.

The general Solution of Equation (1) is given by :

$$y = A\, J_n(x) \ + B\, Y_n(x)$$

where A and B are arbitrary constants, and

J_n is called the

Bessel Function of the First Kind of Order n.

Bessel Functions of the first kind of order n

$$J_n(x) = \frac{x^n}{2^n\,\Gamma(n+1)}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right\}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\,\Gamma(n+k+1)}$$

Γ is called the **Gamma Function**

$$\Gamma(n)=\int_0^\infty t^{n-1}e^{-t}dt \quad for\ n\rangle 0$$

$$\Gamma(n+1)=n\Gamma(n)$$

$$\Gamma(n+1)=n! \ ; \ if\ n=0,1,2,\ldots.where\ \ 0!=1$$

$$\Gamma(n)=\frac{\Gamma(n+1)}{n} \quad for\ n\langle 0$$

Bessel Functions of the first kind of order n

$$J_{-n}(x) = \frac{x^{-n}}{2^{-n}\Gamma(1-n)}\left\{1 - \frac{x^2}{2(2-2n)} + \frac{x^4}{2\cdot 4(2-2n)(4-2n)} - \cdots\right\}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-n}}{k!\,\Gamma(k+1-n)}$$

$$J_{-n}(x) = (-1)^n J_n(x) \qquad n = 0, 1, 2, \ldots$$

For n=0,1 we have

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \cdots$$

$$J_0'(x) = -J_1(x)$$

Y_n is called the *Bessel Function of the Second Kind of Order n*.

$$Y_n(x) = \frac{2}{\pi}\{\ln(x/2)+\gamma\}J_n(x) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(x/2)^{2k-n} - \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\{\Phi(k)+\Phi(n+k)\}\frac{(x/2)^{2k+n}}{k!\,(n+k)!}$$

γ is Euler's Constant and is defined by

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln\ n\right) \approx 0.5772$$

$$\Phi(p) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p},$$

$$\Phi(0) = 0$$

For n=0

$$Y_0(x) = \frac{2}{\pi}\{\ln(x/2) + \gamma\} J_0(x)$$

$$+ \frac{2}{\pi}\left\{\frac{x^2}{2^2} - \frac{x^4}{2^2 4^2}(1+\tfrac{1}{2}) + \frac{x^6}{2^2 4^2 6^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) - \cdots\right\}$$

$$Y_{-n}(x) = (-1)^n Y_n(x) \qquad n = 0, 1, 2, \ldots$$

General Solution of Bessel Differential Equation

$$y = AJ_n(x) + BJ_{-n}(x) \qquad n \neq 0, 1, 2, \ldots$$

$$y = AJ_n(x) + BY_n(x) \qquad \text{all } n$$

$$y = AJ_n(x) + BJ_n(x) \int \frac{dx}{xJ_n^2(x)} \qquad \text{all } n$$

where A and B are arbitrary constants.

Generating Function for $J_n(x)$

$$e^{x(t-1/t)/2} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

Recurrence Formulas forBessel Functions

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

$$J'_n(x) = \tfrac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$$

$$x J'_n(x) = x J_{n-1}(x) - n J_n(x)$$

$$x J'_n(x) = n J_n(x) - x J_{n+1}(x)$$

$$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$$

$$\frac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$$

The functions $Y_n(x)$ satisfy identical relations.

Bessel Functions of Order Equal to Half and Odd Integer

In this case the functions are expressible in terms of sines and cosines.

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$$

$$J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right\}$$

$$J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\frac{3}{x}\sin x + \left(\frac{3}{x^2} - 1\right)\cos x\right\}$$

For further results use the recurrence formula.

Results for $Y_{1/2}(x), Y_{3/2}(x), \ldots$ are obtained from

$$Y_n(x) = \begin{cases} \dfrac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi} & n \neq 0, 1, 2, \ldots \\[2ex] \lim\limits_{p \to n} \dfrac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi} & n = 0, 1, 2, \ldots \end{cases}$$

Bessels Modified Differential Equations

$$x^2y'' + xy' - (x^2 + n^2)y = 0 \qquad n \geqq 0$$

Solutions of this equation are called modified Bessel functions of order n.

Modified Bessels Functions of the First Kind of Order n

$$I_n(x) = i^{-n} J_n(ix) = e^{-n\pi i/2} J_n(ix)$$

$$= \frac{x^n}{2^n \Gamma(n+1)} \left\{ 1 + \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} + \cdots \right\} = \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!\,\Gamma(n+k+1)}$$

Modified Bessels Functions of the First Kind of Order n

$$I_{-n}(x) = i^n J_{-n}(ix) = e^{n\pi i/2} J_{-n}(ix)$$

$$= \frac{x^{-n}}{2^{-n}\Gamma(1-n)}\left\{1 + \frac{x^2}{2(2-2n)} + \frac{x^4}{2\cdot 4(2-2n)(4-2n)} + \cdots\right\} = \sum_{k=0}^{\infty} \frac{(x/2)^{2k-n}}{k!\,\Gamma(k+1-n)}$$

Modified Bessels Functions of the First Kind of Order n

$$I_{-n}(x) = I_n(x) \qquad n = 0, 1, 2, \ldots$$

For $n = 0, 1$, we have

$$I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

$$I_1(x) = \frac{x}{2} + \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \cdots$$

$$I_0'(x) = I_1(x)$$

Modified Bessels Functions of the Second Kind of Order n

$$K_n(x) = \begin{cases} \dfrac{\pi}{2 \sin n\pi}\{I_{-n}(x) - I_n(x)\} & n \neq 0, 1, 2, \ldots \\ \lim\limits_{p \to n} \dfrac{\pi}{2 \sin p\pi}\{I_{-p}(x) - I_p(x)\} & n = 0, 1, 2, \ldots \end{cases}$$

For $n = 0, 1, 2, \ldots,$ L'Hospital's rule yields

Modified Bessels Functions of the Second Kind of Order n

$$K_n(x) = (-1)^{n+1}\{\ln(x/2) + \gamma\} I_n(x) +$$

$$\frac{1}{2}\sum_{k=0}^{n-1} (-1)^k (n-k-1)!\,(x/2)^{2k-n}$$

$$+ \frac{(-1)^n}{2}\sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!\,(n+k)!}\{\Phi(k) + \Phi(n+k)\}$$

Modified Bessels Functions of the Second Kind of Order n

For $n = 0$,

$$K_0(x) = -\{\ln(x/2) + \gamma\} I_0(x) + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots$$

$$K_{-n}(x) = K_n(x) \qquad n = 0, 1, 2, \ldots$$

General Solution of Bessel's Modified Equation

$$y = A I_n(x) + B I_{-n}(x) \qquad n \neq 0, 1, 2, \ldots$$

$$y = A I_n(x) + B K_n(x) \qquad \text{all } n$$

$$y = A I_n(x) + B I_n(x) \int \frac{dx}{x I_n^2(x)} \qquad \text{all } n$$

where A and B are arbitrary constants.

Generating Function for $I_n(x)$

$$e^{x(t+1/t)/2} = \sum_{n=-\infty}^{\infty} I_n(x) t^n$$

Recurrence Formulas for Modified Bessel Functions

$$I_{n+1}(x) = I_{n-1}(x) - \frac{2n}{x} I_n(x)$$

$$I'_n(x) = \tfrac{1}{2}\{I_{n-1}(x) + I_{n+1}(x)\}$$

$$x I'_n(x) = x I_{n-1}(x) - n I_n(x)$$

$$x I'_n(x) = x I_{n+1}(x) + n I_n(x)$$

$$\frac{d}{dx}\{x^n I_n(x)\} = x^n I_{n-1}(x)$$

$$\frac{d}{dx}\{x^{-n} I_n(x)\} = x^{-n} I_{n+1}(x)$$

Recurrence Formulas for Modified Bessel Functions

$$K_{n+1}(x) = K_{n-1}(x) + \frac{2n}{x} K_n(x)$$

$$K_n'(x) = \tfrac{1}{2}\{K_{n-1}(x) + K_{n+1}(x)\}$$

$$x K_n'(x) = -x K_{n-1}(x) - n K_n(x)$$

$$x K_n'(x) = n K_n(x) - x K_{n+1}(x)$$

$$\frac{d}{dx}\{x^n K_n(x)\} = -x^n K_{n-1}(x)$$

$$\frac{d}{dx}\{x^{-n} K_n(x)\} = -x^{-n} K_{n+1}(x)$$

Modified Bessel Functions of Order Equal to Half and Odd Integer

In this case the functions are expressible in terms of hyperbolic sines and cosines.

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$$

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh x$$

$$I_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\cosh x - \frac{\sinh x}{x} \right)$$

Modified Bessel Functions of Order Equal to Half and Odd Integer

$$I_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\sinh x - \frac{\cosh x}{x}\right)$$

$$I_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\left(\frac{3}{x^2}+1\right)\sinh x - \frac{3}{x}\cosh x\right\}$$

$$I_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\left(\frac{3}{x^2}+1\right)\cosh x - \frac{3}{x}\sinh x\right\}$$

For further results use the recurrence formula.

Results for $K_{1/2}(x), K_{3/2}(x),$ are obtained from

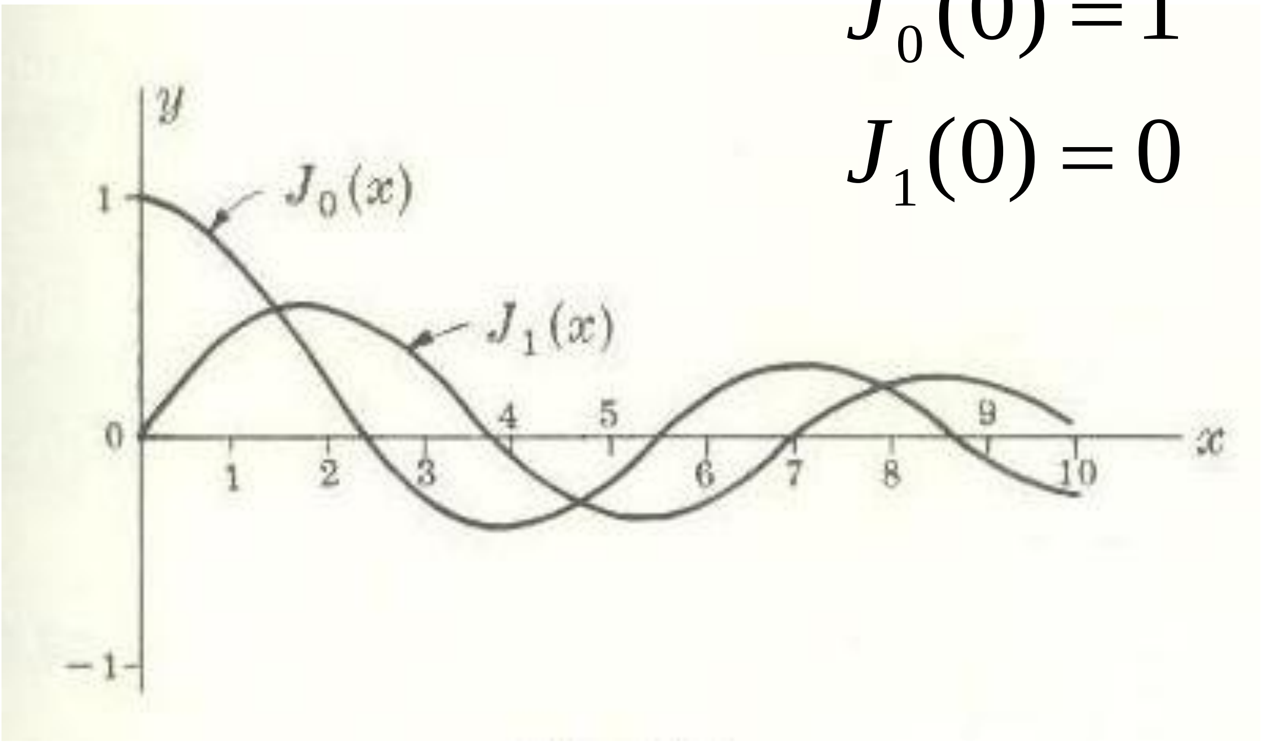
Modified Bessel Functions of Order Equal to Half and Odd Integer

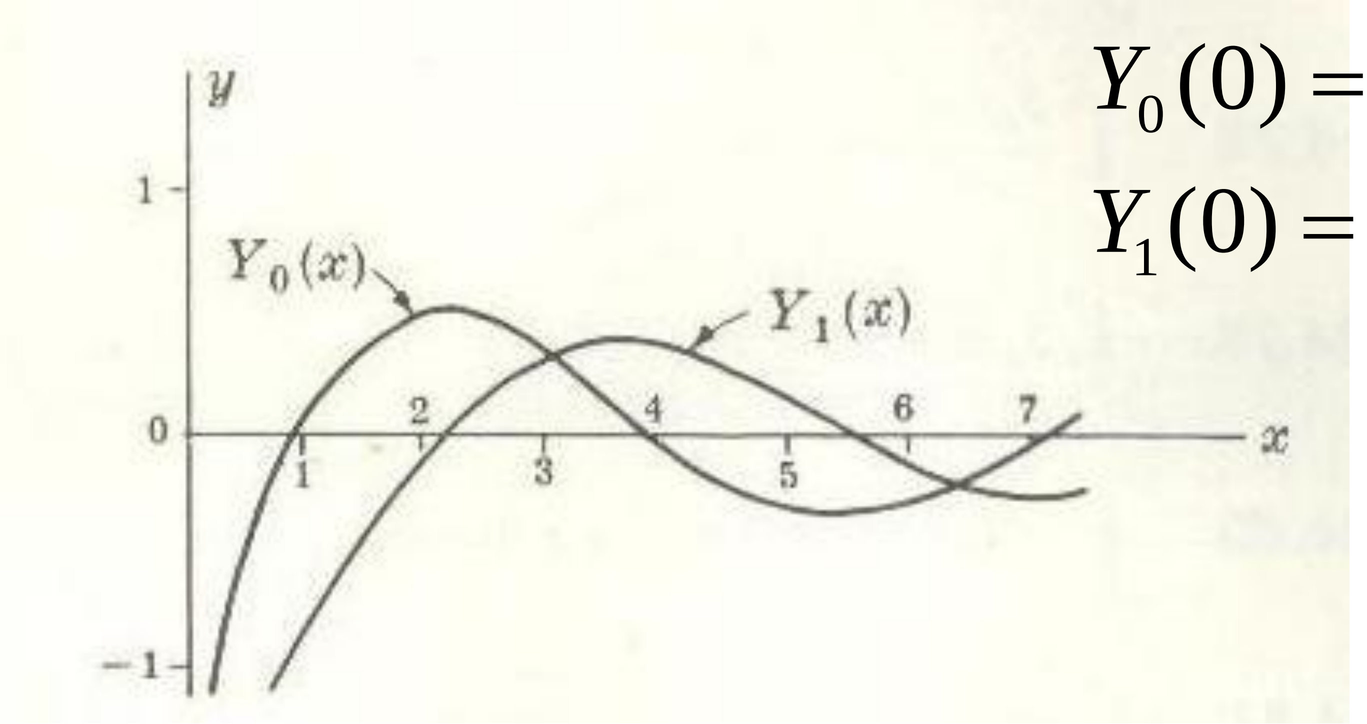
$$K_n(x) = \begin{cases} \dfrac{\pi}{2\sin n\pi}\{I_{-n}(x) - I_n(x)\} & n \neq 0, 1, 2, \ldots \\ \lim\limits_{p\to n} \dfrac{\pi}{2\sin p\pi}\{I_{-p}(x) - I_p(x)\} & n = 0, 1, 2, \ldots \end{cases}$$

Graphs of Bessel Functions

$$J_0(0) = 1$$

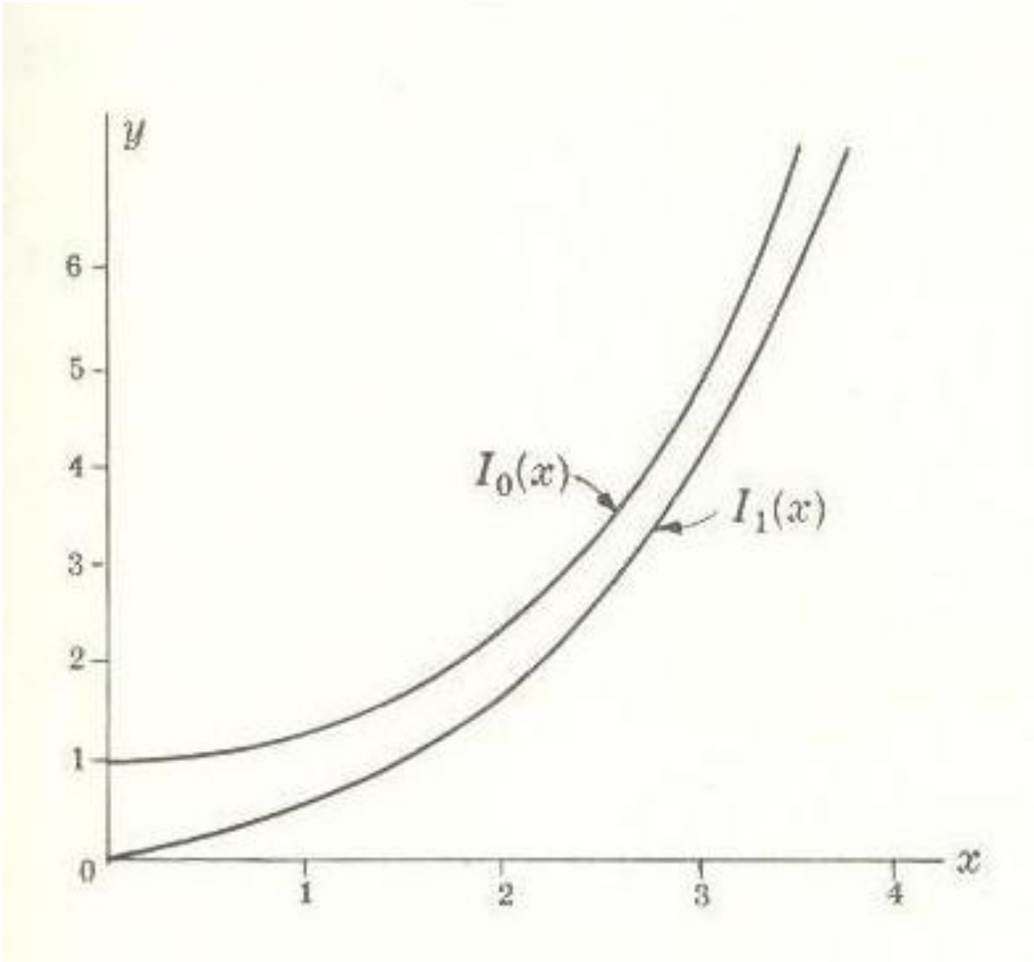
$$J_1(0) = 0$$





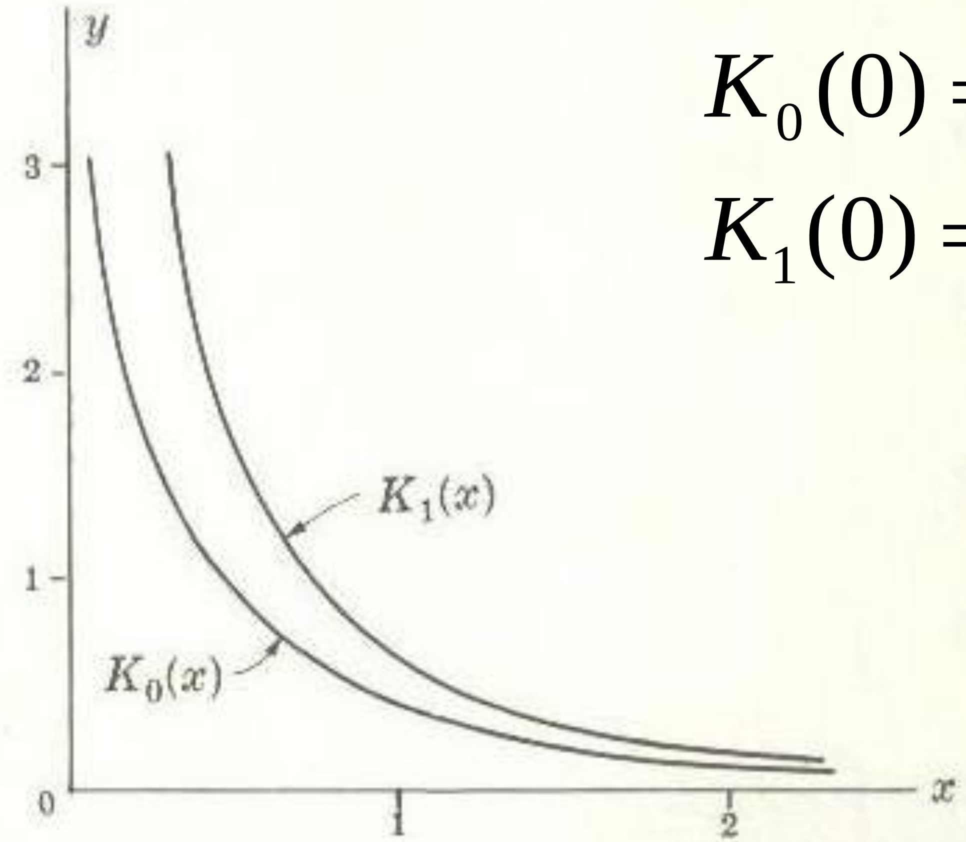
$$Y_0(0) = -\infty$$

$$Y_1(0) = -\infty$$



$$I_1(0) = 0$$

$$I_o(0) = 1$$



$$K_0(0) = +\infty$$

$$K_1(0) = +\infty$$

Indefinite Integrals Involving Bessel Functions

$$\int x\,J_0(x)\,dx = x\,J_1(x)$$

$$\int x^2\,J_0(x)\,dx = x^2\,J_1(x) + x\,J_0(x) - \int J_0(x)\,dx$$

$$\int x^m\,J_0(x)\,dx = x^m\,J_1(x) + (m-1)x^{m-1}\,J_0(x) - (m-1)^2 \int x^{m-2}\,J_0(x)\,dx$$

Indefinite Integrals Involving Bessel Functions

$$\int \frac{J_0(x)}{x^2}\,dx = J_1(x) - \frac{J_0(x)}{x} - \int J_0(x)\,dx$$

$$\int \frac{J_0(x)}{x^m}\,dx = \frac{J_1(x)}{(m-1)^2 x^{m-2}} - \frac{J_0(x)}{(m-1)x^{m-1}} - \frac{1}{(m-1)^2}\int \frac{J_0(x)}{x^{m-2}}\,dx$$

Indefinite Integrals Involving Bessel Functions

$$\int J_1(x)\,dx = -J_0(x)$$

$$\int x\,J_1(x)\,dx = -x\,J_0(x) + \int J_0(x)\,dx$$

$$\int x^m\,J_1(x)\,dx = -x^m\,J_0(x) + m\int x^{m-1}\,J_0(x)\,dx$$

Indefinite Integrals Involving Bessel Functions

$$\int \frac{J_1(x)}{x}\, dx = -J_1(x) + \int J_0(x)\, dx$$

$$\int \frac{J_1(x)}{x^m}\, dx = -\frac{J_1(x)}{mx^{m-1}} + \frac{1}{m}\int \frac{J_0(x)}{x^{m-1}}\, dx$$

$$\int x^n J_{n-1}(x)\, dx = x^n J_n(x)$$

$$\int x^{-n} J_{n+1}(x)\, dx = -x^{-n} J_n(x)$$

Definite Integrals Involving Bessel Functions

$$\int_0^\infty e^{-ax} J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}$$

$$\int_0^\infty e^{-ax} J_n(bx)\,dx = \frac{(\sqrt{a^2+b^2}-a)^n}{b^n\sqrt{a^2+b^2}} \qquad n > -1$$

$$\int_0^\infty \cos ax\, J_0(bx)\,dx = \begin{cases} \dfrac{1}{\sqrt{a^2-b^2}} & a > b \\ 0 & a < b \end{cases}$$

Definite Integrals Involving Bessel Functions

$$\int_0^\infty J_n(bx)\,dx = \frac{1}{b} \qquad n > -1$$

$$\int_0^\infty \frac{J_n(bx)}{x}\,dx = \frac{1}{n} \qquad n = 1, 2, 3, \ldots$$

$$\int_0^\infty e^{-ax} J_0(b\sqrt{x})\,dx = \frac{e^{-b^2/4a}}{a}$$

A General Differential Equation Having Bessel Functions as Solutions

Many differential equations occur in practice that are not of the standars form but whose solutions can be written in terms of Bessel functions.

A General Differential Equation Having Bessel Functions as Solutions

The differential equation

$$y'' + \frac{1-2a}{x} y' + \left[\left(bcx^{c-1}\right)^2 + \frac{a^2 - p^2c^2}{x^2} \right] y = 0$$

has the solution

$$y = x^a Z_p(bx^c)$$

Where Z stands for J and Y or any linear combination of them, and a, b, c, p are constants.

Example

Solve y''+9xy=0

Solution:

$$1-2a=0;$$

$$(bc)^2=9;$$

$$2(c-1)=1;$$

$$a^2-p^2c^2=0$$

From these equations we find

$$a = 1/2;$$

$$c = 3/2;$$

$$b = 2;$$

$$p = a/c = 1/3$$

Then the solution of the equation is

$$y = x^{1/2} Z_{1/3}(2x^{3/2})$$

This means that the general solution of the equation is

$$y = x^{1/2}[AJ_{1/3}(2x^{3/2}) + BY_{1/3}(2x^{3/2})$$

where A and B are constants

A General Differential Equation Having Bessel Functions as Solutions

The differential equation

$$x^2 y'' + x(a + 2bx^p)y' + \left[c + dx^{2q} + b(a + p - 1)x^p + b^2 x^{2p}\right]y = 0$$

If $(1 - a^2) \geq 4c \ and \ d \ and \ \ p \ or \ q \ is \ not \ zero$

has the solution

$$y = x^{\alpha} e^{-\beta\, x^p}\left[AZ_{\upsilon}(\lambda x^q) + BZ_{-\upsilon}(\lambda x^q)\right]$$

$$\alpha = \frac{1-a}{2};$$

$$\beta = \frac{b}{p};$$

$$\lambda = \frac{\sqrt{|d|}}{q};$$

$$\upsilon = \frac{\sqrt{(1-a)^2 - 4c}}{2q}$$

υ	d	Z_{υ}	$Z_{-\upsilon}$
$\neq 0$	$d\rangle 0$	J_{υ}	$J_{-\upsilon}$
$= 0$	$d\rangle 0$	J_{υ}	Y_{υ}
$\neq 0$	$d\langle 0$	I_{υ}	$I_{-\upsilon}$
$= 0$	$d\langle 0$	I_{υ}	K_{υ}

A General Differential Equation Having Bessel Functions as Solutions

The differential equation

$$\frac{d}{dx}\left(x^{r} \frac{dy}{dx} \right) + \left(ax^{s} + bx^{r-2} \right) y = 0$$

If $(1 - r^{2}) \geq 4b \ and \ s \rangle r - 2 \ \ or \ b = 2$

has the solution

$$y = x^{\alpha}\left[AZ_{\upsilon}(\lambda x^{\gamma}) + BZ_{-\upsilon}(\lambda x^{\gamma}) \right]$$

$$\alpha = \frac{1-r}{2};$$

$$\gamma = \frac{2-r+s}{2};$$

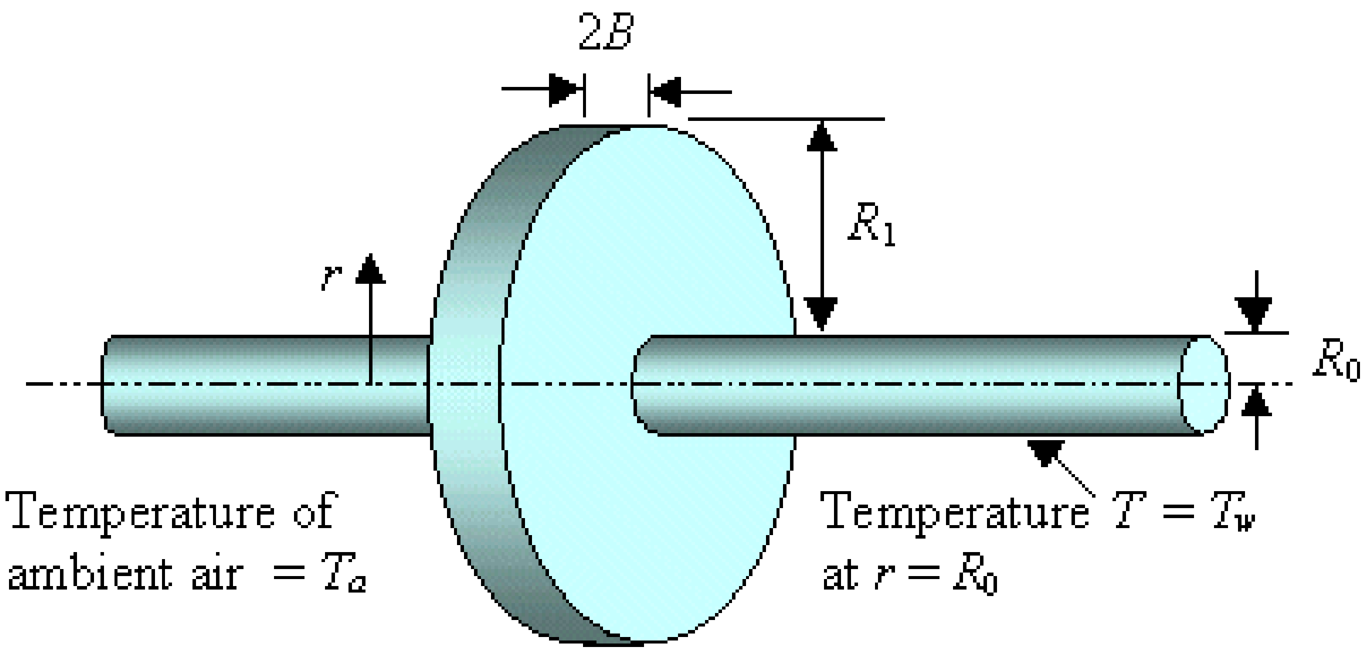
$$\lambda = \frac{2\sqrt{|a|}}{2-r+s};$$

$$\upsilon = \frac{\sqrt{(1-r)^2 - 4b}}{2-r+s}$$

υ	a	Z_{υ}	$Z_{-\upsilon}$
$\neq 0$	$a \rangle 0$	J_{υ}	$J_{-\upsilon}$
$= 0$	$a \rangle 0$	J_{υ}	$Y_{-\upsilon}$
$\neq 0$	$a \langle 0$	I_{υ}	$I_{-\upsilon}$
$= 0$	$a \langle 0$	I_{υ}	K_{υ}

Problem

A pipe of radius R_0 has a circular fin of radius R_1 and thickness $2B$ on it (as shown in the figure below). The outside wall temperature of the pipe is T_w and the ambient air temperature is T_a. Neglect the heat loss from the edge of the fin (of thickness $2B$). Assume heat is transferred to the ambient air by surface convection with a constant heat transfer coefficient h.


2B
R1
r
R0
Temperature of
ambient air = Ta
Temperature T = Tw
at r = R0

- a) Starting with a shell thermal energy balance, derive the differential equation that describes the radial temperature distribution in the fin.
- b) Obtain the radial temperature distribution in the circular fin.
- c) Develop an expression for the total heat loss from the fin.

Solution

From a thermal energy balance over a thin cylindrical ring of width Δr in the circular fin, we get

Rate of Heat In - Out + Generation = Accumulation

The accumulation term (at steady-state) and the generation term will be zero. So,

$$(2\pi r 2B q_r)\ \Big|_r - (2\pi r 2B q_r)\ \Big|_{r+\Delta r} - 2(2\pi r \Delta r) h (T - T_a) = 0$$

where h is the (constant) heat transfer coefficient for surface convection to the ambient air and q_r is the heat flux for conduction in the radial direction.

Dividing by $4\pi B\,\Delta r$ and taking the limit as Δr tends to zero,

$$\lim_{\Delta r \to 0} \frac{(r q_r)\ \Big|_{r+\Delta r} - (r q_r)\ \Big|_r}{\Delta r} = -\frac{h}{B} r (T - T_a)$$

$$\frac{d}{dr}(rq_r) = -\frac{h}{B}r(T - T_a)$$

If the thermal conductivity *k* of the fin material is considered constant, on substituting Fourier's law we get

$$\frac{d}{dr}(r\frac{dT}{dr}) = \frac{h}{kB}r(T - T_a)$$

Let the dimensionless excess temperature be denoted by $\theta = (T - T_a)/(T_w - T_a)$. Then,

$$\frac{d}{dr}(r\frac{d\theta}{dr}) - \frac{h}{kB}r\theta = 0$$

$$\frac{d}{dx}\left(x^r \frac{dy}{dx} \right) + \left(ax^s + bx^{r-2}\right)y = 0$$

$$r = 1; s = 1; b = 0; a = -h/(kB)$$

$$(1-r^2) \geq 4b \quad 1-1^2 = 0 = 4(0)$$

$$and \quad s \rangle r - 2$$

$$1 \rangle 1 - 2$$

$$\alpha = 0;$$

$$\gamma = 1;$$

$$\lambda = \sqrt{|a|}$$

$$\upsilon = 0$$

$$\theta = x^0 \left\lfloor AZ_0(\sqrt{|a|}x) + BZ_{-0}(\sqrt{|a|}x) \right\rfloor$$

$$\theta = \left\lfloor AI_0(\sqrt{|a|}x) + BK_0(\sqrt{|a|}x) \right\rfloor$$