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On Some Injective Modules In $\sigma[M]$

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Dedicated to Professor Robert Wisbauer on his 65th birthday

Abstract. In this paper, we study the notions (strongly) soc-injective, (strongly) simple-injective and (strongly) mininjective modules in $\sigma[M]$. For any module N in $\sigma[M]$, N is strongly mininjective in $\sigma[M]$ if and only if it is strongly simple-injective in $\sigma[M]$. A module M is locally Noetherian if and only if every strongly simple-injective module in $\sigma[M]$ is strongly soc-injective. We also characterize Noetherian QF-modules.

1. Introduction

Let M be any R-module. Any R-module N is *generated by* M or M-*generated* if there exists an epimorphism $M^{(\Lambda)} \longrightarrow N$ for some index set Λ. An R-module N is said to be *subgenerated* by M if N is isomorphic to a submodule of an M-generated module. We denote by $\sigma[M]$ the full subcategory of the right R-modules whose objects are all right R-modules subgenerated by M.

Let M be a module and let N and T be in $\sigma[M]$. N is called *soc-T-injective* if any R-homomorphism $f : \mathrm{Soc}(T) \to N$ extends to T. Equivalently, for any semisimple submodule K of T, any homomorphism $f : K \to N$ extends to T. A module $N \in \sigma[M]$ is called *soc-quasi-injective in* $\sigma[M]$ if N is soc-N-injective. N is called *soc-injective in* $\sigma[M]$ if N is soc-M-injective. N is called *strongly soc-injective in* $\sigma[M]$ if N is soc-T-injective for all $T \in \sigma[M]$.

According to Harada [7], if M and N are modules, M is called *simple-N-injective* if, for every submodule L of N, every homomorphism $\gamma : L \longrightarrow M$ with $\gamma(L)$ simple extends to N. If $N = R$, M is called *simple-injective*, and if $M = N$, M is called *simple-quasi-injective*. Dually, M is called *min-N-injective* if, for every simple submodule L of N, every homomorphism $\gamma : L \to M$ extends to N. If $N = R$, M is called *mininjective*, and if $M = N$, M is called *min-quasi-injective*. Let $T \in \sigma[M]$. T is called *strongly simple-injective* in $\sigma[M]$ if T is simple-N-injective for all $N \in \sigma[M]$, and T is called *strongly mininjective* in $\sigma[M]$, if T is min-N-injective for all $N \in \sigma[M]$ (see [2]).

Throughout this article, all rings are associative and have an identity, and all modules are unitary right R-modules. Let M be an R-module. For a direct summand N of M we write $N \leq_d M$ and for an essential submodule N of M, $N \leq_e M$. Let $\widehat{N}$ be the M-injective hull of N in $\sigma[M]$. A module N in $\sigma[M]$ is called *M-singular* (or *singular in* $\sigma[M]$) if $N \cong L/K$ for an $L \in \sigma[M]$ and $K \leq_e L$ (see [6]). Every module $N \in \sigma[M]$ contains a largest M-singular submodule which is denoted by $Z_M(N)$. If $Z_M(N) = 0$, then N is called *non-M-singular*. We will use $\mathrm{Soc}(K)$ to indicate the socle of any module K.

In Section 2, we prove that, for any finitely generated module T in $\sigma[M]$, direct sums of soc-T-injective modules in $\sigma[M]$ is soc-T-injective if and only if $\mathrm{Soc}(T)$ is finitely generated. Also it is proven that if $N \in \sigma[M]$ is soc(N)-lifting, then any module K in $\sigma[M]$ is soc-N-injective if and only if K is N-injective.

In Section 3, we consider the strongly soc-injective modules in $\sigma[M]$. Semiartinian and Noetherian QF-modules are characterized in terms of strongly soc-injective modules in $\sigma[M]$. For example, any module M is semiartinian if and only if every strongly soc-injective module in $\sigma[M]$ is injective in $\sigma[M]$ (quasi-continuous). Let M be a finitely generated self-projective module. Then M is a Noetherian QF-module if and only if every strongly soc-injective module in $\sigma[M]$ is projective in $\sigma[M]$ if and only if M is a self-generator, $\mathrm{Soc}(M) \leq_e M$ and every projective module in $\sigma[M]$ is strongly soc-injective in $\sigma[M]$ if and only if $M/\mathrm{Soc}(M)$ has finite length and M is a self-generator strongly soc-injective in $\sigma[M]$. In this section we also characterize GCO-modules and cosemisimple modules in terms of strongly soc-injective modules in $\sigma[M]$.

In Section 4, we consider soc-injective modules. Let S and R be any rings and let M be a left S-, a right R-bimodule. We prove that if M_R is soc-injective, then $l_S(T_1 \cap T_2) = l_S(T_1) + l_S(T_2)$ for all semisimple submodules T_1 and T_2 of M_R while $l_S(A \cap B) = l_S(A) + l_S(B)$ for all semisimple submodules A and all submodules B of M_R in the case where $S = End_R(M)$.

In the last section, it is shown that the notions of strongly mininjective and strongly simple-injective coincide. We also prove that any module M is locally Noetherian if and only if every strongly simple-injective module in $\sigma[M]$ is strongly soc-injective, and that if M is finitely generated self-projective, then M is a Noetherian QF-module if and only if every strongly simple-injective module in $\sigma[M]$ is projective in $\sigma[M]$.

2. Soc-Injective Modules in $\sigma[M]$

Theorem 2.1. *Let M be a module.*

(1) *Let $N \in \sigma[M]$ and $\{M_i : i \in I\}$ a family of right R-modules in $\sigma[M]$. Then the direct product $\prod_{i\in I} M_i$ is soc-N-injective if and only if M_i is soc-N-injective for all $i \in I$.*

(2) *Let T, N and $K \in \sigma[M]$ with $K \leq N$. If T is soc-N-injective, then T is soc-K-injective.*

(3) *Let T, N and $K \in \sigma[M]$ with $T \cong N$. If T is soc-K-injective, then N is soc-K-injective.*
(4) *Let $N \in \sigma[M]$ and $\{A_i : i \in I\}$ a family of right R-modules in $\sigma[M]$. Then N is soc-$\oplus_{i\in I} A_i$-injective if and only if N is soc-A_i-injective for all $i \in I$.*
(5) *Let M be a projective module in $\sigma[M]$. Any module $N \in \sigma[M]$ is soc-injective if and only if N is soc-P-injective for every M-generated projective module P in $\sigma[M]$.*
(6) *Let T, N and $K \in \sigma[M]$ with $N \leq_d T$. If T is soc-K-injective, then N is soc-K-injective.*
(7) *If A, B and $N \in \sigma[M]$, $A \cong B$ and N is soc-A-injective, then N is soc-B-injective.*

Proof. Clear. □

The next corollary is an immediate consequence of Theorem 2.1.

Corollary 2.2.

(1) *If $N \in \sigma[M]$, then a finite direct sum of soc-N-injective modules in $\sigma[M]$ is again soc-N-injective. In particular, a finite direct sum of soc-injective* (*strongly soc-injective*) *modules in $\sigma[M]$ is again soc-injective* (*strongly soc-injective*).
(2) *A direct summand of soc-quasi-injective* (*soc-injective, strongly soc-injective*) *module in $\sigma[M]$ is again soc-quasi-injective* (*soc-injective, strongly soc-injective*).

Proposition 2.3. *Suppose $N \in \sigma[M]$ is a soc-quasi-injective module.*

(1) (Soc-C_2) *If K and L are semisimple submodules of N, $K \cong L$ and $K \leq_d N$, then $L \leq_d N$.*
(2) (Soc-C_3) *Let K and L be semisimple submodules of N with $K \cap L = 0$. If $K \leq_d N$ and $L \leq_d N$, then $K \oplus L \leq_d N$.*

Proof. (1) Since $K \cong L$, and K is soc-N-injective, being a direct summand of the soc-quasi-injective module N, L is soc-N-injective . If $i : L \to N$ is the inclusion map, the identity map $id_L : L \to L$ has an extension $\eta : N \to L$ such that $\eta i = id_L$, and so $L \leq_d N$.

(2) Then both K and L are soc-N-injective. Thus the semisimple module $K \oplus L$ is soc-N-injective, and so a direct summand of N. □

Proposition 2.4. *For $N \in \sigma[M]$, the following are equivalent:*

(1) *Every module in $\sigma[M]$ is soc-N-injective.*
(2) *Every semisimple module in $\sigma[M]$ is soc-N-injective.*
(3) *$Soc(N) \leq_d N$.*

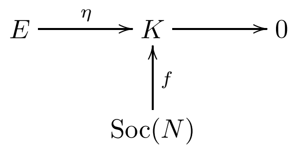
Proof. Straightforward. □

Theorem 2.5. *For a projective module $N \in \sigma[M]$, the following are equivalent:*

(1) *Every quotient of a soc-N-injective module in $\sigma[M]$ is soc-N-injective.*
(2) *Every quotient of an injective module in $\sigma[M]$ is soc-N-injective.*
(3) *Soc(N) is projective in $\sigma[M]$.*

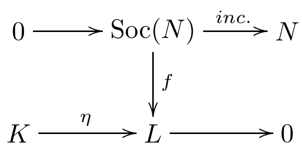
Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Consider the following diagram



where E and K are in $\sigma[M]$, η is an epimorphism and f any homomorphism. By Cartan and Eilenberg [4], we may assume that E is injective in $\sigma[M]$. Since K is soc-N-injective, f can be extended to $g : N \to K$. Since N is projective in $\sigma[M]$, g can be lifted to $\tilde{g} : N \to E$ such that $\eta\tilde{g} = g$. Now define $\tilde{f} : \mathrm{Soc}(N) \to E$ by $\tilde{f} = \tilde{g}|_{\mathrm{Soc}(N)}$. Clearly, $\eta\tilde{f} = f$. Hence Soc(N) is projective in $\sigma[M]$.

(3) $\Rightarrow$ (1) Let $K \in \sigma[M]$ be soc-N-injective. Assume $\eta : K \to L$ is an epimorphism. We want to show that L is soc-N-injective. Consider the following diagram



Since Soc(N) is projective, f can be lifted to $g : \mathrm{Soc}(N) \to K$. Since K is soc-injective, g can be extended to $\tilde{g} : N \to K$. Clearly $\eta\tilde{g} : N \to L$ extends f. □

Corollary 2.6. *The following are equivalent for a projective module M in $\sigma[M]$:*

(1) *Every quotient of a soc-injective module in $\sigma[M]$ is soc-injective in $\sigma[M]$.*
(2) *Every quotient of an injective module in $\sigma[M]$ is soc-injective in $\sigma[M]$.*
(3) *Every semisimple submodule of a projective module in $\sigma[M]$ is projective in $\sigma[M]$.*
(4) *Soc(M) is projective in $\sigma[M]$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) By Theorem 2.5.

(3) $\Rightarrow$ (4) Since M is projective in $\sigma[M]$, Soc(M) is projective in $\sigma[M]$.

4) $\Rightarrow$ (3) If P is a projective module in $\sigma[M]$, then it is a direct summand of a direct sum of finitely generated submodules of $M^{(\mathbb{N})}$ by [9, 18.4]. Then Soc(P) is a direct summand of a direct sum of socles of finitely generated submodules of $M^{(\mathbb{N})}$. Since Soc(M) is projective in $\sigma[M]$, then Soc(P) is projective in $\sigma[M]$. Hence (3) follows. □

Theorem 2.7. *Let $T \in \sigma[M]$ be finitely generated. Then the following are equivalent:*

(1) *Direct sums of soc-T-injective modules in $\sigma[M]$ is soc-T-injective.*
(2) $\operatorname{Soc}(T)$ *is finitely generated.*

Proof. (1) $\Rightarrow$ (2) Let $\operatorname{Soc}(T) = \oplus_{i \in I} S_i$ where each S_i is a simple submodule of T. Let $\widehat{S}_i$ be the injective hull of S_i in $\sigma[M]$, $i \in I$, and $\iota : \oplus_{i\in I} S_i \to \oplus_{i\in I}\widehat{S}_i$ be the inclusion map. Since $\oplus_{i\in I}\widehat{S}_i$ is soc-T-injective, ι can be extended to an R-homomorphism $\hat{\iota} : T \to \oplus_{i\in I}\widehat{S}_i$. Since T is finitely generated, $\hat{\iota}(T) \leq \oplus_{i=1}^{n}\widehat{S}_i$, for some positive integer n. Therefore $\operatorname{Soc}(T) \leq \oplus_{i=1}^{n}\widehat{S}_i$ implies that $\operatorname{Soc}(T)$ is finitely generated.

(2) $\Rightarrow$ (1) Let $E = \oplus_{i\in I} E_i$ be a direct sum of soc-T-injective modules in $\sigma[M]$ and $f : \operatorname{Soc}(T) \to E$ be an R-homomorphism. Since $\operatorname{Soc}(T)$ is finitely generated, $f(\operatorname{Soc}(T)) \leq \oplus_{i=1}^{n} E_i$, for some positive integer n. Since $\oplus_{i=1}^{n} E_i$ is soc-T-injective, f can be extended to an R-homomorphism $\hat{f} : T \to T$. □

Corollary 2.8. *Let M be finitely generated. Then the following are equivalent:*

(1) *Direct sums of soc-injective modules in $\sigma[M]$ are soc-injective.*
(2) $\operatorname{Soc}(M)$ *is finitely generated.*

Corollary 2.9. *The following are equivalent:*

(1) *Direct sums of soc-T-injective modules in $\sigma[M]$ are soc-T-injective for every cyclic R-module T in $\sigma[M]$.*
(2) *Finitely generated R-modules in $\sigma[M]$ are finite dimensional.*

Definition 2.10. Let X be a submodule of a module M. We say that *$Soc(M)$ respects X* if there exists a direct summand A of M contained in X such that $X = A \oplus B$ and $B \leq \operatorname{Soc}(M)$. M is called *$Soc(M)$-lifting* if $\operatorname{Soc}(M)$ respects every submodule of M.

Proposition 2.11. *Let $N \in \sigma[M]$. If N is $\operatorname{Soc}(N)$-lifting, then any module K in $\sigma[M]$ is soc-N-injective if and only if K is N-injective.*

Proof. Assume that a module $K \in \sigma[M]$ is soc-N-injective. Let L be any submodule of N, $i_2 : L \to N$ the inclusion map and $f : L \to K$ any homomorphism. By hypothesis, L has a decomposition $L = A \oplus B$ such that A is a direct summand of N and $B \leq \operatorname{Soc}(N)$. $N = A \oplus A'$ for some submodule A' of N. Then $L = A \oplus (L \cap A')$ and $L \cap A'$ is semisimple. Let $i_1 : L \cap A' \to L$ be the inclusion map and $f|_{L\cap A'} : L \cap A' \to K$. Since K is soc-N-injective, there exists a homomorphism $g : N \to K$ such that $gi_2i_1 = f|_{L\cap A'}$. Now define $h : N \to K$ by $h(a + a') = f(a) + g(a')$ $(a \in A, a' \in A')$. Then $hi_2 = f$. □

Corollary 2.12. [11, Lemma 2.14] *If $R/\operatorname{Soc}(R_R)$ is semisimple, then a right R-module M is soc-injective in Mod-R if and only if M is injective.*

Proof. $R/\operatorname{Soc}(R_R)$ is semisimple if and only if $\operatorname{Soc}(R_R)$ respects every right ideal of R [11, Theorem 2.3]. Hence by Proposition 2.11, the result holds. □

Clearly if Soc(M) respects every submodule of M, then $M/Soc(M)$ is semisimple. We don't know if the converse is true or not.

3. Strongly soc-injective modules in $\sigma[M]$

Theorem 3.1. *Let $N \in \sigma[M]$. The following are equivalent:*

(1) *N is strongly soc-injective in $\sigma[M]$.*
(2) *N is soc-$\widehat{N}$-injective.*
(3) *$N = E \oplus T$, where E is injective in $\sigma[M]$ and T has zero socle.*

Moreover, if N has a nonzero socle, then E can be taken to have essential socle.

Proof. (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (3) If $\mathrm{Soc}(N) = 0$, we are done. Assume that $\mathrm{Soc}(N) \neq 0$, and consider the following diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Soc}(N) & \xrightarrow{\ i\ } & \widehat{\mathrm{Soc}(N)} \\ & & \downarrow{\scriptstyle \iota} & & \\ & & N & & \end{array}$$

where ι and i are inclusion maps. Since N is soc-$\widehat{N}$-injective, N is soc-$\widehat{\mathrm{Soc}(N)}$-injective. So, there exists an R-homomorphism $\sigma : \widehat{\mathrm{Soc}(N)} \to N$, which extends ι. Since $\mathrm{Soc}(N) \leq_e \widehat{\mathrm{Soc}(N)}$, σ is an embedding of $\widehat{\mathrm{Soc}(N)}$ in N. If we write $E = \sigma(\widehat{\mathrm{Soc}(N)})$, then $N = E \oplus T$ for some submodule T of N. Clearly, E is injective and T has zero socle.
(3) $\Rightarrow$ (1) This is clear, since modules with zero socle are strongly soc-injective in $\sigma[M]$ and finite direct sum of strongly soc-injective modules are strongly soc-injective in $\sigma[M]$.

For the last statement of the theorem, then $\sigma(\mathrm{Soc}(N)) \leq_e E$. On the other hand, $\mathrm{Soc}(E) = \mathrm{Soc}(N) = \sigma(\mathrm{Soc}(N)) \leq_e E$ implies that $\mathrm{Soc}(E) \leq_e E$. □

Corollary 3.2. *Let $N \in \sigma[M]$ be a module with essential socle. Then the following are equivalent:*

(1) *N is strongly soc-injective in $\sigma[M]$.*
(2) *N is injective in $\sigma[M]$.*

A module M is called *locally Noetherian* if every finitely generated submodule of M is Noetherian. It is well known that M is locally Noetherian if and only if every direct sum of M-injective modules is M-injective [9, 27.3], if and only if every (countable) direct sum of M-injective hulls of simple modules (in $\sigma[M]$) is M-injective ([9, 27.3] and [6, 2.5]).

Theorem 3.3. *The following are equivalent for a module M:*

(1) *M is locally Noetherian.*

(2) *Every direct sum of strongly soc-injective modules in $\sigma[M]$ is strongly soc-injective in $\sigma[M]$.*

Proof. (1) $\Rightarrow$ (2) Let $\{M_i\}_{i \in I}$ be a family of strongly soc-injective modules in $\sigma[M]$. By Theorem 3.1, for each $i \in I$, write $M_i = E_i \oplus T_i$ where E_i is injective in $\sigma[M]$ and $\mathrm{Soc}(T_i) = 0$. If $E = \oplus_{i \in I} E_i$ and $T = \oplus_{i \in I} T_i$, then $\oplus_{i \in I} M_i = E \oplus T$ with $\mathrm{Soc}(T) = 0$. Since M is locally Noetherian, E is M-injective, that is injective in $\sigma[M]$, and by Theorem 3.1, $\oplus_{i \in I} M_i$ is strongly soc-injective in $\sigma[M]$.

(2) $\Rightarrow$ (1) In order to prove that M is locally Noetherian, we only need to show that if $K_1, K_2, \ldots$ are simple modules (in $\sigma[M]$), then $\oplus_{i=1}^{\infty} \widehat{K_i}$ is injective in $\sigma[M]$, where $\widehat{K_i}$ is the M-injective hull of K_i. Since $\oplus_{i=1}^{\infty} \widehat{K_i}$ is strongly soc-injective in $\sigma[M]$ with essential socle, by Corollary 3.2, $\oplus_{i=1}^{\infty} \widehat{K_i}$ is injective in $\sigma[M]$. $\square$

Proposition 3.4. *If $N \in \sigma[M]$ is strongly soc-injective in $\sigma[M]$, then every semisimple submodule K of N is essential in a direct summand of N.*

Proof. This is clear if $\mathrm{Soc}(N) = 0$. If $\mathrm{Soc}(N) \neq 0$, then by Theorem 3.1, $N = \widehat{\mathrm{Soc}(N)} \oplus T$ with $\mathrm{Soc}(T) = 0$. Then $K \leq_e L \leq_d \widehat{\mathrm{Soc}(N)}$ for some submodule L of N. $\square$

M is called *CESS* if every closure of every semisimple submodule of M is a direct summand of M. By Theorem 3.1, if $N \in \sigma[M]$ is strongly soc-injective in $\sigma[M]$, then $N = E \oplus T$ with $E = \widehat{\mathrm{Soc}(N)}$ and $\mathrm{Soc}(T) = 0$, and by [5], if T is E-injective, then N is a CESS-module. In particular, if T is non-M-singular, then T is E-injective and so N is a CESS-module.

Proposition 3.5. *Let $N \in \sigma[M]$ be $N = E \oplus T$ with $E = \widehat{\mathrm{Soc}(N)}$, $\mathrm{Soc}(T) = 0$ and T is E-injective. If S is a semisimple submodule of N, then every closure in N, of S is injective in $\sigma[M]$.*

Proof. By the above remark, if K is a closure of S in N, then K is a direct summand of N, and by Corollary 2.2 (2), K is strongly soc-injective in $\sigma[M]$. Let K' be a closure of S in E. Then K' is a direct summand of E and so is injective in $\sigma[M]$. Now we consider the following diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & S & \xrightarrow{\;i\;} & K' \\ & & \downarrow{\scriptstyle \iota} & & \\ & & K & & \end{array}$$

where ι and i are inclusion maps. Since K is strongly soc-injective in $\sigma[M]$, there exists a homomorphism $\sigma : K' \to K$ which extends ι. Since $S \leq_e K'$, σ is an embedding of K' in K, and so $S \leq_e \sigma(K') \leq_e K$, since $\sigma(K')$ is injective in $\sigma[M]$, it is a direct summand of K, and so $\sigma(K') = K$ is injective in $\sigma[M]$. □

A module M is called *semiartinian* if every nonzero homomorphic image of M has essential socle. Equivalently, every nonzero homomorphic image of M has nonzero socle. M is semiartinian if and only if every module in $\sigma[M]$ is semiartinian (see [6, 3.12]).

Theorem 3.6. *The following are equivalent for a module M:*

(1) *M is semiartinian.*
(2) *Every strongly soc-injective module in $\sigma[M]$ is injective in $\sigma[M]$.*
(3) *Every strongly soc-injective module in $\sigma[M]$ is quasi-continuous.*

Proof. (1) ⇒ (2) Since M is semiartinian, $\mathrm{Soc}(N) \leq_e N$ for every module $N \in \sigma[M]$. By Corollary 3.1, (2) holds.

(2) ⇒ (3) Clear.

(3) ⇒ (1) Let N be a proper submodule of M. We claim that $\mathrm{Soc}(M/N) \neq 0$. If $\mathrm{Soc}(M/N) = 0$, let X/N be an arbitrary nonzero submodule of M/N. By hypothesis, $(X/N) \oplus (M/N)$ is quasi-continuous. By [8, Corollary 2.14], X/N is M/N-injective and hence $X/N \leq_d M/N$. This means that M/N is semisimple, a contradiction. Hence M is semiartinian. □

If M is a Noetherian injective cogenerator in $\sigma[M]$, then it is called a *Noetherian Quasi-Frobenius* (QF)*-module*. For a finitely generated quasi-projective module M, M is Noetherian QF-module if and only if every injective module in $\sigma[M]$ is projective in $\sigma[M]$ if and only if M is a self-generator and every projective module in $\sigma[M]$ is injective in $\sigma[M]$ by [9, 48.14].

Proposition 3.7. *Let M be a finitely generated self-projective module. Then the following are equivalent:*

(1) *M is a Noetherian QF-module.*
(2) *Every strongly soc-injective module in $\sigma[M]$ is projective in $\sigma[M]$.*

Proof. (1) ⇒ (2) If M is a Noetherian QF-module, then M is Artinian by [9, 48.14]. By Theorem 3.6, every strongly soc-injective module in $\sigma[M]$ is injective in $\sigma[M]$, and hence projective in $\sigma[M]$ by [9, 48.14].

(2) ⇒ (1) Clear. □

Observe that if $\mathrm{Soc}(M) = 0$, then every projective module in $\sigma[M]$ has zero socle by [9, 18.4(1)], and hence strongly soc-injective in $\sigma[M]$. On the other hand we have the following result by Corollary 3.2 and the above remark.

Proposition 3.8. *Let M be a finitely generated self-projective module. Then the following are equivalent:*

(1) *M is a Noetherian QF-module.*
(2) *M is a self-generator,* $\mathrm{Soc}(M) \leq_e M$ *and every projective module in $\sigma[M]$ is strongly soc-injective in $\sigma[M]$.*

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (1) Let P be a nonzero projective module in $\sigma[M]$. Then P is strongly soc-injective in $\sigma[M]$. By Theorem 3.1, $P = E \oplus T$ with E injective in $\sigma[M]$ and $\mathrm{Soc}(T) = 0$. On the other hand, P is a direct summand of a direct sum of nonzero finitely generated submodules M_i of $M^{(\mathbb{N})}$. Since every M_i has essential socle, $\mathrm{Soc}(P) \leq_e P$. Therefore $P = E$, and hence P is injective in $\sigma[M]$. Since M is a self-generator, the proof is completed by [9, 48.14]. □

Any module M is called $\sum$*-injective* if the direct sum of any number of copies of M is injective.

Proposition 3.9. *Let M be a projective module in $\sigma[M]$. Then the following are equivalent:*

1. *Every projective M-generated module in $\sigma[M]$ is strongly soc-injective in $\sigma[M]$.*
2. *$M = E \oplus T$ where E is $\sum$-injective in $\sigma[M]$ and $\mathrm{Soc}(T) = 0$.*

Proof. (1) $\Rightarrow$ (2) If $\mathrm{Soc}(M) = 0$, we are done. Assume $\mathrm{Soc}(M)$ is nonzero. Since M is projective, it follows from Theorem 3.1 that $M = E \oplus T$ where E is injective in $\sigma[M]$ with essential socle and $\mathrm{Soc}(T) = 0$. Since for any ordinal number α, $E^{(\alpha)}$ is projective in $\sigma[M]$ and M-generated, $E^{(\alpha)}$ is strongly soc-injective with essential socle. Therefore by Corollary 3.2, $E^{(\alpha)}$ is injective in $\sigma[M]$. Hence E is $\sum$-injective in $\sigma[M]$.

(2) $\Rightarrow$ (1) By (2), $M^{(\Lambda)} = E^{(\Lambda)} \oplus T^{(\Lambda)}$ for any ordinal number Λ. Since $E^{(\Lambda)}$ is injective in $\sigma[M]$, $M^{(\Lambda)}$ is strongly soc-injective in $\sigma[M]$ by Theorem 3.1. Let P be a projective M-generated module in $\sigma[M]$. Then P is isomorphic to a direct summand of $M^{(\Lambda)}$ for some Λ. Since every direct summand of strongly soc-injective module in $\sigma[M]$ is strongly soc-injective in $\sigma[M]$, P is strongly soc-injective in $\sigma[M]$. □

Proposition 3.10. *Let $N \in \sigma[M]$ be a strongly soc-injective module. If $N/\mathrm{Soc}(N)$ is finite dimensional (Noetherian, Artinian, respectively), then $N = T \oplus S$, where T is finite dimensional (Noetherian, Artinian, respectively) and S is semisimple injective in $\sigma[M]$.*

Proof. By Theorem 3.1, $N = E \oplus K$ with E is injective in $\sigma[M]$ and $\mathrm{Soc}(K) = 0$. Now, $N/\mathrm{Soc}(N) \cong E/\mathrm{Soc}(E) \oplus K$. So both $E/\mathrm{Soc}(E)$ and K are finite dimensional (Noetherian, Artinian, respectively). By [3, Corollary 3], $E = L \oplus S$ with L finite dimensional and S semisimple. If $E/\mathrm{Soc}(E)$ is Noetherian (Artinian), then by [3, Lemma 4 and Proposition 5], $E = L \oplus S$ where L is Noetherian (Artinian) and S is semisimple. Consequently, $N = T \oplus S$ with S semisimple injective in $\sigma[M]$ and $T = K \oplus L$ finite dimensional (Noetherian, Artinian, respectively). □

Corollary 3.11. *Let M be a finitely generated self-projective module in $\sigma[M]$. Then the following are equivalent:*

1. *M is a Noetherian QF-module.*

(2) *$M/\mathrm{Soc}(M)$ has finite length and M is a self-generator strongly soc-injective in $\sigma[M]$.*

Proof. (2) $\Rightarrow$ (1) By Proposition 3.10, $\mathrm{Soc}(M) \leq_e M$. Then by Corollary 3.2, M is injective in $\sigma[M]$. Again by Proposition 3.10, M is Noetherian. By [9, 48.14], M is a Noetherian QF-module.

(1) $\Rightarrow$ (2) By [9, 48.14], $M/\mathrm{Soc}(M)$ has finite length and M is a self-generator. Since M is projective in $\sigma[M]$, by [9, 48.14], M is injective in $\sigma[M]$ and hence M is strongly soc-injective in $\sigma[M]$. □

Lemma 3.12. *Let $N \in \sigma[M]$ be semisimple. The following are equivalent:*

(1) *N is injective in $\sigma[M]$.*
(2) *N is strongly soc-injective in $\sigma[M]$.*
(3) *N is soc-K-injective for every factor module K of M.*

Proof. (1) $\Leftrightarrow$ (2) By Corollary 3.2.

(1) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{i} & M \\ & & \big\downarrow{\scriptstyle f} & & \\ & & N & & \end{array}$$

where $L \leq M$ and $f : L \longrightarrow N$ is any homomorphism. Then we have the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & L/kerf & \xrightarrow{\bar{i}} & M/kerf \\ & & \big\downarrow{\scriptstyle \alpha} & & \\ & & f(L) & & \\ & & \big\downarrow{\scriptstyle \iota} & & \\ & & N & & \end{array}$$

where α is an isomorphism and ι is the inclusion map.

Since N is soc-$M/Kerf$-injective and $L/Kerf$ is semisimple, there exists a homomorphism $g : M/Kerf \longrightarrow N$ such that $g\bar{i} = \iota\alpha$. Then the homomorphism $h = g\pi$ extends f where $\pi : M \longrightarrow M/Kerf$ is the natural epimorphism. □

A module M is called *cosemisimple* (*or a V-module*) if every simple module (in $\sigma[M]$) is M-injective. Clearly, M is cosemisimple if and only if every simple module is strongly soc-injective in $\sigma[M]$.

Proposition 3.13. *The following are equivalent for a module M:*

(1) *Every semisimple module in $\sigma[M]$ is strongly soc-injective in $\sigma[M]$.*
(2) *Every semisimple module in $\sigma[M]$ is soc-K-injective for every factor module K of M.*

(3) *Every module in* $\sigma[M]$ *is strongly soc-injective in* $\sigma[M]$.
(4) *Every module in* $\sigma[M]$ *is soc-*K*-injective for every factor module* K *of* M.
(5) *Every semisimple module in* $\sigma[M]$ *is injective in* $\sigma[M]$.
(6) M *is locally Noetherian and cosemisimple.*

Proof. (5) $\Leftrightarrow$ (6) by [6, 15.5].

(1) $\Leftrightarrow$ (3) By Proposition 2.4.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (5) By Lemma 3.12.

(4) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (4) By (2) $\Leftrightarrow$ (3). □

A module M is called *generalized cosemisimple* (or a *GCO-module*) if every simple singular module is M-injective or M-projective. Equivalently, every M-singular simple module is M-injective by [6, 16.4].

By adopting the above proof we have the following proposition. Note that (5) $\Leftrightarrow$ (6) of Proposition 3.14 is well known from [6, 16.16].

Proposition 3.14. *The following are equivalent for a module* M*:*

(1) *Every semisimple* M*-singular module is strongly soc-injective in* $\sigma[M]$.
(2) *Every semisimple* M*-singular module in* $\sigma[M]$ *is soc-*K*-injective for every factor module* K *of* M.
(3) *Every* M*-singular module in* $\sigma[M]$ *is strongly soc-injective in* $\sigma[M]$.
(4) *Every* M*-singular module in* $\sigma[M]$ *is a direct sum of an injective module in* $\sigma[M]$ *and a module with zero socle.*
(5) *Every* M*-singular semisimple module in* $\sigma[M]$ *is injective in* $\sigma[M]$.

If M *is self-projective, then they are equivalent to*

(6) M *is a GCO-module and* $M/\mathrm{Soc}(M)$ *is locally Noetherian.*

4. When M is soc-injective

Proposition 4.1. *Let* M *be a module. The following are equivalent:*

(1) M *is soc-injective.*
(2) *If* $\mathrm{Soc}(M) = X \oplus Y$ *and* $\gamma : X \longrightarrow M$ *is an* R*-homomorphism, then there exists* $c : M \longrightarrow M$ *such that* $\gamma(x) = c(x)$ *for all* $x \in X$ *and* $c(Y) = 0$.
(3) *If* $X \subseteq \mathrm{Soc}(M)$ *and* $\gamma : X \longrightarrow M$ *is an* R*-homomorphism, then there exists* $c : M \longrightarrow M$ *such that* $\gamma(x) = c(x)$ *for all* $x \in X$.

If M *is finitely generated and self-projective in* $\sigma[M]$*, then* (1)–(3) *are equivalent to*

(4) *If* K *is semisimple,* P *is projective* M*-generated in* $\sigma[M]$*,* Q *is a finitely generated projective* M*-generated in* $\sigma[M]$*,* $\iota : K \longrightarrow P$ *is a monomorphism and* $f : K \longrightarrow Q$ *is an* R*-homomorphism, then* f *can be extended to an* R*-homomorphism* $\tilde{f} : P \longrightarrow Q$.

Proof. (1) ⇒ (2) Let $\mathrm{Soc}(M) = X \oplus Y$ and $\gamma : X \longrightarrow M$ be an R-homomorphism. Define the homomorphism $\tilde{\gamma} : X \oplus Y \longrightarrow M$ by $x+y \mapsto \gamma(x)$ $(x \in X, y \in Y)$. Since M is soc-M-injective, $\tilde{\gamma}$ can be extended to the homomorphism $c : M \longrightarrow M$. Let $x \in X$. Then $c(x) = \tilde{\gamma}(x) = \gamma(x)$. Let $y \in Y$. Then $c(y) = \tilde{\gamma}(y) = \gamma(0) = 0$. Thus $c(Y) = 0$.

(2) ⇒ (3) ⇒ (1) and (4) ⇒ (1) are clear.

(1) ⇒ (4) Since M is soc-injective, M is soc-P-injective. Clearly, Q is isomorphic to a direct summand of $M^{(n)}$, for some positive integer n. Therefore Q is soc-P-injective by Theorem 2.1. Thus f can be extended to $\tilde{f} : P \longrightarrow Q$. □

Proposition 4.2. *Let M be a soc-injective module. Then the following holds.*

(1) *M satisfies (Soc-C_2),*
(2) *M satisfies (Soc-C_3).*

Proof. Take $N = M$ in Proposition 2.3. □

Let R and S be rings with identity and M a left S-, a right R-bimodule. For any $X \subseteq M$ and any $T \subseteq S$ denote $l_S(X) = \{s \in S \mid sX = 0\}$ and $r_M(T) = \{m \in M \mid Tm = 0\}$.

Note that if M is a right R-module then M is a left $End_R(M)$-module. If $l_S(A \cap B) = l_S(A) + l_S(B)$ for all submodules A and B of M_R, where $S = End_R(M)$, M is called an *Ikeda-Nakayama* module [10]. Note that every quasi-injective module is an Ikeda-Nakayama module [10, Lemma 1]. For a soc-injective module we have the following result.

Proposition 4.3. *Let S and R be any rings and M a left S-, a right R-bimodule. If M_R is soc-injective, then*

(1) *$l_S(T_1 \cap T_2) = l_S(T_1) + l_S(T_2)$ for all semisimple submodules T_1, T_2 of M_R.*
(2) *If Sk is a simple left S-module $(k \in M)$, then $\mathrm{Soc}(kR)$ is zero or simple.*
(3) *$r_M l_S(\mathrm{Soc}(M)) = \mathrm{Soc}(M) \Leftrightarrow r_M l_S(K) = K$ for all semisimple submodule K of M_R.*

Proof. (1) By [10, Lemma 1].

(2) Assume Sk $(k \in M)$ is a simple left S-module and $\mathrm{Soc}(kR)$ is nonzero. Let y_1R and y_2R be simple submodules of M_R with $y_i \in kR, 1 \le i \le 2$. If $y_1R \cap y_2R = 0$, then by (1), $l_S(y_1) + l_S(y_2) = S$ and so $l_S(y_1) = l_S(y_2) = l_S(k)$, since $y_i \in kR$ and $l_S(k)$ is a maximal left ideal of S. Thus $l_S(k) = S$, a contradiction, hence $\mathrm{Soc}(kR)$ is simple.

(3) Assume that $r_M l_S(\mathrm{Soc}(M)) = \mathrm{Soc}(M)$ and let K be a semisimple submodule of M_R. We claim that K is essential in $r_M l_S(K)$. If $K \cap xR = 0$ for some $x \in r_M l_S(K)$, then by (1), $l_S(K \cap xR) = l_S(K) + l_S(xR) = S = l_S(xR)$ since $x \in r_M l_S(K) \le r_M l_S(\mathrm{Soc}(M)) = \mathrm{Soc}(M)$ and $l_S(K) \le l_S(xR)$. Then $x = 0$. Hence $K \le_e r_M l_S(K) \le r_M l_S(\mathrm{Soc}(M)) = \mathrm{Soc}(M)$. It follows that $K = r_M l_S(K)$. The converse is clear. □

Proposition 4.4. *Let M be a right R-module and $S = End_R(M)$. Then the following are equivalent:*

(1) $r_M l_S(K) = K$ *for all semisimple submodules K of M_R.*
(2) $r_M[l_S(K) \cap Sa] = K + r_M(a)$ *for all semisimple submodules K of M_R and all $a \in S$.*

Proof. (1) ⇒ (2) Clearly, $K + r_M(a) \leq r_M[l_S(K) \cap Sa]$. Let $x \in r_M[l_S(K) \cap Sa]$ and $y \in l_S(aK)$. Then $yaK = 0$ and $ya \in Sa \cap l_S(K)$, so $yax = 0$ and $y \in l_S(ax)$. Thus $l_S(aK) \leq l_S(ax)$, and so $ax \in r_M l_S(ax) \leq r_M l_S(aK)$. Since Soc$(M)$ is fully invariant, aK is a semisimple submodule of M_R. By (1), $ax \in aK$. Hence $ax = ak$ for some $k \in K$ and so $x - k \in r_M(a)$. This means that $x \in r_M(a) + K$.

(2) ⇒ (1) The case when $a = 1_S$. □

Proposition 4.5. *Let M be a right R-module and $S = End_R(M)$. If M_R is strongly soc-injective in $\sigma[M]$, then $l_S(A \cap B) = l_S(A) + l_S(B)$ for all semisimple submodules A and all submodules B of M_R.*

Proof. Let $x \in l_S(A \cap B)$ and define $\psi : A + B \longrightarrow M_R$ by $\psi(a + b) = xa$ for all $a \in A$ and $b \in B$. This induces an R-homomorphism $\tilde{\psi} : (A + B)/B \longrightarrow M_R$ in the obvious way. Since (A+B)/B is semisimple and M_R is strongly soc-injective in $\sigma[M]$, $\tilde{\psi}$ can be extended to an R-homomorphism $\varphi : M/B \longrightarrow M$. Now let $\pi : M \longrightarrow M/B$ be the natural epimorphism. Let denote $s = \varphi\pi \in S$. Let $b \in B$. Then $sb = \varphi\pi(b) = \varphi(b+B) = 0$. For any $a \in A$, $(x-s)a = xa-sa = xa-\varphi\pi(a) = 0$. It follows that $x = (x - s) + s \in l_S(A) + l_S(B)$. □

5. Strongly simple-injective modules in $\sigma[M]$

Theorem 5.1. *The following are equivalent for $N \in \sigma[M]$:*

(1) *N is strongly mininjective in $\sigma[M]$.*
(2) *N is strongly simple-injective in $\sigma[M]$.*
(3) *Every homomorphism from a finitely generated semisimple submodule K of any module $T \in \sigma[M]$ into N extends to T.*
(4) *Every homomorphism γ from a submodule K of any module $T \in \sigma[M]$ into N, with $\gamma(K)$ finitely generated semisimple, extends to T.*

Proof. (4) ⇒ (3) ⇒ (1) Clear.

(1) ⇒ (2) Let L be a submodule of N and $\gamma : L \longrightarrow K$ a homomorphism with $\gamma(L)$ simple. If $T = Ker\gamma$, then γ induces an embedding $\tilde{\gamma} : L/T \longrightarrow K$ defined by $\tilde{\gamma}(x + T) = \gamma(x)$ for all $x \in L$. Since K is strongly mininjective and L/T is simple, $\tilde{\gamma}$ extends to a homomorphism $\overline{\gamma} : N/T \longrightarrow K$. If $\eta : N \longrightarrow N/T$ is the natural epimorphism, the homomorphism $\overline{\gamma}\eta : N \longrightarrow K$ is an extension of γ, for if $x \in L$, $(\overline{\gamma}\eta)(x) = \overline{\gamma}(x + T) = \tilde{\gamma}(x + T) = \gamma(x)$, as required.

(2) $\Rightarrow$ (4) Let T be any module in $\sigma[M]$, K a submodule of T, $\gamma : K \to N$ a homomorphism with $\gamma(K)$ finitely generated semisimple and consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{i} & T \\ & & {\scriptstyle\gamma}\big\downarrow & & \\ & & N & & \end{array}$$

Write $\gamma(K) = \oplus_{i=1}^{n} S_i$ where each S_i is simple. Let $\pi_i \oplus_{i=1}^{n} S_i \to S_i$ be the canonical projection, $1 \leq i \leq n$, and consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{i} & T \\ & & {\scriptstyle\pi_i\gamma}\big\downarrow & & \\ & & N & & \end{array}$$

Since N is strongly simple-injective in $\sigma[M]$, for each i, $1 \leq i \leq n$, there exists a homomorphism $\gamma_i : T \to N$ such that $\gamma_i(x) = \pi_i\gamma(x)$, for all $x \in K$. Now, define the map $\hat{\gamma} : T \to N$ by $\hat{\gamma}(x) = \sum_{i=1}^{n} \gamma_i(x)$. Then $\hat{\gamma}(x) = \gamma(x)$ for all $x \in K$. □

Hence we have the following implications:

$$\begin{array}{rcl} \text{soc-}N\text{-injective} & \Longrightarrow & \text{min-}N\text{-injective} \\ \text{simple-}N\text{-injective} & \Longrightarrow & \text{min-}N\text{-injective} \\ \text{strongly mininjective} & \Longleftrightarrow & \text{strongly simple-injective} \end{array}$$

Min-N-injective modules need not be soc-N-injective (see [1, Example 4.5] and [1, Example 4.15]), and strongly simple-injective modules need not be strongly soc-injective (see [2, Remark 2.4] and [1]).

Proposition 5.2. (1) *Let $N \in \sigma[M]$ and $\{M_i : i \in I\}$ be a family of modules in $\sigma[M]$. Then the direct product $\prod_{i\in I} M_i$ is min-N-injective if and only if each M_i is min-N-injective, $i \in I$. In particular, $\prod_{i\in I} M_i$ is strongly simple-injective if and only if each M_i is strongly simple-injective, $i \in I$.*

(2) *If $\{M_i : i \in I\}$ is a family of modules in $\sigma[M]$, then the direct sum $\oplus_{i\in I} M_i$ is strongly simple-injective if and only if each M_i is strongly simple-injective, $i \in I$.*

(3) *A direct summand of a strongly simple-injective module is strongly simple-injective.*

(4) *Let M be projective. M is strongly simple-injective if and only if every M-generated projective module $N \in \sigma[M]$ is strongly simple-injective.*

Proof. Routine. □

Note 5.3. As in Corollary 2.6, for a projective module M, every quotient of a simple-injective module in $\sigma[M]$ is simple-injective if and only if $Soc(M)$ is projective in $\sigma[M]$.

Corollary 5.4. *Let $N \in \sigma[M]$ such that $Soc(N)$ is finitely generated (in particular, if M is finite dimensional), then the following are equivalent:*

(1) *N is strongly mininjective in $\sigma[M]$.*
(2) *N is strongly simple-injective in $\sigma[M]$.*
(3) *N is strongly soc-injective in $\sigma[M]$.*

Moreover, if in addition $Soc(N) \leq_e N$, then each of the above conditions is equivalent to

(4) *M is injective.*

Proof. By Theorem 5.1 and Corollary 3.2. □

Theorem 5.5. *The following are equivalent for $N \in \sigma[M]$:*

(1) *N is strongly simple-injective in $\sigma[M]$.*
(2) *N is min-$\widehat{M}$-injective.*
(3) *N is min-$\widehat{S}$-injective for every simple module $S \in \sigma[M]$.*
(4) *N is min-$\widehat{S}$-injective for every simple submodule S of N.*

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) Clear.

(4) ⇒ (1) Let $T \in \sigma[M]$, $\gamma : K \to N$ a non-zero homomorphism with $\gamma(K)$ simple, and consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \gamma(K) & \xrightarrow{\ i\ } & \widehat{\gamma(K)} \\ & & {\scriptstyle i}\downarrow & & \\ & & N & & \end{array}$$

where i is the inclusion map. Since N is min-$\widehat{\gamma(K)}$-injective, there exists an embedding $\sigma : \widehat{\gamma(K)} \to N$ such that $\sigma\gamma(x) = \gamma(x)$ for every $x \in K$. Now, the map γ may be viewed as a map from K into an M-injective submodule of N, and hence has an extension $\widehat{\gamma} : T \to N$. □

Corollary 5.6. *If $N \in \sigma[M]$ is strongly simple-injective, then every simple submodule of N is essential in an M-injective direct summand of N.*

Proof. Let S be a simple submodule of N and consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & S & \xrightarrow{\ i\ } & \widehat{S} \\ & & {\scriptstyle i}\downarrow & & \\ & & N & & \end{array}$$

where i is the inclusion map. Since N is min-$\widehat{S}$-injective and $S \leq_e \widehat{S}$, there exists an embedding σ of $\widehat{S}$ in N such that $\sigma(x) = x$ for all $x \in S$. If $E = \sigma(\widehat{S}) \cong \widehat{S}$, then $S \leq_e E \leq_d N$. □

Proposition 5.7. *The following are equivalent for M:*

(1) *M is locally Noetherian.*
(2) *Every strongly simple-injective module in $\sigma[M]$ is strongly soc-injective.*

Proof. (1) $\Rightarrow$ (2) Suppose M is locally Noetherian, and N is strongly simple-injective in $\sigma[M]$. Write $Soc(N) = \oplus_{i\in I} S_i$, where each S_i is simple, $i \in I$. By Corollary 5.6, each $S_i \leq_e E_i \leq_d N$, where E_i is M-injective, $i \in I$. Since M is locally Noetherian, $E = \oplus_{i\in I} E_i$ is M-injective and hence E is a direct summand of N, and so $N = E \oplus T$, with $Soc(T) = 0$. By Theorem 3.1, N is strongly soc-injective in $\sigma[M]$.

(2) $\Rightarrow$ (1) Let $\{K_i\}_{i\in I}$ be a family of simple modules in $\sigma[M]$. Consider $\widehat{K_i}$ for each $i \in I$. Therefore every $\widehat{K_i}$ is strongly simple-injective in $\sigma[M]$. Then by Proposition 5.2(2), $E = \oplus_{i=1}^{\infty} \widehat{K_i}$ is strongly simple-injective in $\sigma[M]$, and hence strongly soc-injective in $\sigma[M]$. Since E has essential socle, by Corollary 3.2, E is injective in $\sigma[M]$. Therefore M is locally Noetherian by [9, 27.3]. □

Proposition 5.8. *Let M be a finitely generated self-projective module. Then the following are equivalent:*

(1) *M is a Noetherian QF-module.*
(2) *Every strongly simple-injective module in $\sigma[M]$ is projective in $\sigma[M]$.*

Proof. (1) $\Rightarrow$ (2) By Proposition 5.7 and Proposition 3.7.
(2) $\Rightarrow$ (1) By Proposition 3.7. □

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